Partial Differential Equations 3150

Fall 2009

- **Sample Midterm Exam 2**
  - Problem 1. Do either 1a or 1b.
    - Problem 1a. **(CH3. Finite String: Fourier Series Solution)**
    - Problem 1b. **(CH3. d'Alembert's Solution: Finite String)**
  - Problem 2. **(CH3. Heat Conduction in a Bar)**
  - Problem 3. **(CH4. Rectangular Membrane)**
  - Problem 4. **(CH4. Steady-State Heat Conduction on a Disk)**
- **Sample Midterm Exam 3, which is the Final Exam**
  - Problem 1. Selected from Exams 1 and 2.
  - Problem 2. Selected from Exams 1 and 2.
  - Problem 3. Selected from Exams 1 and 2.
  - Problem 4. **(CH4. Poisson Problem)**
  - Problem 5. **(CH4. Heat Equation)**
  - Problem 6. **(CH7. Fourier Transform: Infinite Rod)**

Use this page to start your solution. Attach extra pages as needed, then staple.

Partial Differential Equations 3150

Sample Midterm Exam 2

Exam Date: Tuesday, 1 December 2009

Instructions: This exam is timed for 50 minutes. You will be given double time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1a. (CH3. Finite String: Fourier Series Solution)

(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x,t) &=& c^2 u_{xx}(x,t), & 0 < x < L, & t > 0, \\ u(0,t) &=& 0, & & t > 0, \\ u(L,t) &=& 0, & & t > 0, \\ u(x,0) &=& 0, & 0 < x < L, & \\ u_t(x,0) &=& g(x), & 0 < x < L. & \end{cases}$$

(b) [25%] Display an explicit formula for the Fourier coefficients which contains the symbols L, $g(x)$.

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1b. (CH3. d'Alembert's Solution: Finite String)

Let $f(x) = \begin{cases} 0.3x & 0 \le x \le 0/5, \\ 0.3(1-x) & 0.5 < x \le 1. \end{cases}$ and define $g(x) = 0$ on $0 \le x \le 1$. Assumed is d'Alembert's solution to the vibrating string problem on $0 < x < 1$, $t > 0$, which is the formula

$$u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Assume $L = 1$ and $c = 1/\pi$.

(a) [25%] Define f^*, g^* as appropriate periodic extensions of f and g, respectively.

(b) [75%] Display a piecewise-defined formula for $u(x, \pi/3)$ on $0 < x < 1$.

Use this page to start your solution. Attach extra pages as needed, then staple.

2. (CH3. Heat Conduction in a Bar)

Consider the heat conduction problem in a laterally insulated bar of length 1 with one end at zero Celsius and the other end at 100 Celsius. The initial temperature along the bar is given by function $f(x)$.

$$\begin{cases} u_t & = & c^2 u_{xx}, & 0 < x < 1, & t > 0, \\ u(0,t) & = & 0, & & t > 0, \\ u(1,t) & = & 100, & & t > 0, \\ u(x,0) & = & f(x), & 0 < x < 1. & \end{cases}$$

(a) [25%] Find the steady-state temperature $u_1(x)$.

(b) [50%] Solve the bar problem with zero Celsius temperatures at both ends, but $f(x)$ replace by $f(x) - u_1(x)$. Call the answer $u_2(x,t)$.

(c) [25%] Explain why $u(x,t) = u_1(x) + u_2(x,t)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

3. (CH4. Rectangular Membrane)
Consider the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) &=& c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a, \quad 0<y<b, \quad t>0, \\ u(x,y,t) &=& 0 & \text{on the boundary} \qquad\qquad , \\ u(x,y,0) &=& f(x,y), & 0<x<a, \quad 0<y<b, \\ u_t(x,y,0) &=& g(x,y), & 0<x<a, \quad 0<y<b. \end{cases}$$

Solve the problem for $a=b=1$, $c=1/\pi$, $f(x,y)=0$, $g(x,y)=1$.

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4. (CH4. Steady-State Heat Conduction on a Disk)
Consider the problem

$$\begin{cases} u_{rr}(r,\theta) + \dfrac{1}{r}u_r(r,\theta) + \dfrac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, \quad 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(a,\theta) = f(\theta), \quad 0 < \theta < 2\pi. \end{cases}$$

Solve for $u(r,\theta)$ when $a = 1$ and $f(\theta) = 100\,\mathbf{pulse}(\theta, 0, \pi)$, that is, $f(\theta) = 100$ on $0 \le \theta < \pi$, $f(\theta) = 0$ on $\pi \le \theta < 2\pi$.

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Name________________________________

Partial Differential Equations 3150

Sample Midterm Exam 3 $\equiv$ Final Exam
Final Exam Date: Friday, 18 December 2009
Alternate Final Exam Date: Thursday-Friday, 10-11 December 2009

Instructions: This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

Problems 1,2,3: Selected problem types from the first two midterm exams. See the sample exams for midterms 1 and 2, for specific examples.

4. (CH4. Poisson Problem)
Solve for $u(x,y)$ in the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} & = & \sin(\pi x)\sin(\pi y), & 0 < x < 1, \quad 0 < y < 1, \\ u(x,0) & = & 4\sin(\pi x), & 0 < x < 1, \\ u(x,y) & = & 0 & \text{on the other 3 boundary edges.} \end{cases}$$

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5. (CH4. Heat Equation)
Let $f(x) = x\,\mathbf{pulse}(x, 0, 1)$, that is, $f(x) = x$ for $0 \le x < 1$ and $f(x) = 0$ elsewhere. Solve the insulated rod heat conduction problem

$$\left\{\begin{array}{lcll} u_t(x,t) & = & u_{xx}(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) & = & 0, & \qquad\qquad\quad\; t > 0, \\ u(1,t) & = & 0, & \qquad\qquad\quad\; t > 0, \\ u(x,0) & = & f(x) + \sin(\pi x), & 0 < x < 1. \end{array}\right.$$

Hint: Split the problem into two problems with answers u_1, u_2, such that $u = u_1 + u_2$. Done the right way, $u_1(x,t) = x$ on $0 < x < 1$. The second solution u_2 can be found by Fourier's method, i.e., insert exponential scale factors.

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6. (CH7. Fourier Transform: Infinite Rod)
Let $f(x) = \mathbf{pulse}(x, -2, 2)$, that is, $f(x) = 1$ for $|x| < 2$ and $f(x) = 0$ elsewhere on $-\infty < x < \infty$.

(a) [75%] Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \qquad\quad -\infty < x < \infty. \end{cases}$$

(b) [25%] Evaluate the limit of $u(x,t)$ at $x = \infty$ and $x = -\infty$.

Use this page to start your solution. Attach extra pages as needed, then staple.