Partial Differential Equations 3150

Final Exam

Exam Date: Friday, April 25, 2014

Instructions: This exam is timed for 120 minutes. You may be given extra time to complete the exam (10-15 extra minutes). No calculators, notes, tables or books. Problems use topics from the required textbook which were covered in lectures. Details count 3/4, answers count 1/4.

1. (Chapters H1-H2. Heat Conduction in a Bar)

150

Considered is the heat conduction problem

$$\begin{cases} u_t &= \frac{1}{4}u_{xx}, & 0 < x < 8, \quad t > 0, \\ u(0,t) &= 20, & t > 0, \\ u(8,t) &= -20, & t > 0, \\ u(x,0) &= f(x), & 0 < x < 8. \end{cases} \tag{1}$$

It represents a laterally insulated uniform bar of length 8 with one end held at 20 Celsius and the other end held at -20 Celsius, and initial temperature $f(x)$.

A **1(a)** [30%] Show the details for finding the the steady-state temperature $u_1(x)$.

A **1(b)** [40%] Show the details for the solution $u_2(x,t)$ of the ice-pack ends bar problem

$$\begin{cases} u_t &= \frac{1}{4}u_{xx}, & 0 < x < 8, \quad t > 0, \\ u(0,t) &= 0, & t > 0, \\ u(8,t) &= 0, & t > 0, \\ u(x,0) &= \sum_{n=1}^{100}(-1)^n \sin(2n\pi x/8), & 0 < x < 8. \end{cases}$$

A **1(c)** [30%] Superposition implies $u(x,t) = u_1(x,t) + u_2(x,t)$ is a solution of (1) with $f(x) = u_1(x) + u_2(x,0)$. Show the details of an answer check for the solution $u(x,t)$.

$f(x) = u_1(x) + u_2(x,0)$

Excellent

Use this page to start your solution. Attach any extra pages, then re-staple.

Problem 1:

$$\begin{cases} u_t = \frac{1}{4} u_{xx}, & 0<x<8,\ t>0, \\ u(0,t) = 20, & t>0, \\ u(8,t) = -20, & t>0, \\ u(x,0) = f(x), & 0<x<8. \end{cases}$$

1a) Find the steady-state temperature $u_1(x)$

$\cancel{u_t}^{0} = \frac{1}{4} u_{xx} \Rightarrow u_{xx} = 0 \Rightarrow u = c_1 + c_2 x$

$u(0) = 20 = c_1 + c_2(0) \Rightarrow c_1 = 20$

$u(8) = -20 = 20 + c_2(8) \Rightarrow c_2 = -5$

$$\underline{u_1(x) = 20 - 5x} \checkmark$$

1b) Find $u_2(x,t)$ of the ice-pack ends bar problem

$$\begin{cases} u_t = \frac{1}{4} u_{xx} & 0<x<8,\ t>0, \\ u(0,t) = 0 & t>0, \\ u(8,t) = 0, & t>0, \\ u(x,0) = \sum_{n=1}^{100} (-1)^n \sin\left(\frac{2n\pi}{8} x\right), & 0<x<8. \end{cases}$$

$u = XT$

$$\frac{X T'}{\frac{1}{4} X T} = \frac{\frac{1}{4} X'' T}{\frac{1}{4} X T} \Rightarrow \frac{T'}{\frac{1}{4} T} = \frac{X''}{X} = -\lambda \quad \begin{matrix} \nearrow X'' + \lambda X = 0 \\ \searrow T' + \frac{1}{4}\lambda T = 0 \end{matrix}$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(8) = 0 \end{cases}$$

$\lambda = 0$ is possible but trivial

$\lambda < 0$ doesn't work

$r^2 + \lambda = 0$ $\quad \lambda > 0$

$r = \sqrt{-\lambda}$ $\quad$ ok

$X = a_1 \cos(\sqrt{\lambda} x) + a_2 \sin(\sqrt{\lambda} x)$

$X(0) : 0 = a_1 \cos(0) + a_2 \cancel{\sin(0)}^{0}$

$a_1 = 0$

$X(8) : 0 = a_2 \sin(\sqrt{\lambda}(8))$

$8\sqrt{\lambda} = n\pi$ ✓

$\sqrt{\lambda} = \frac{n\pi}{8}, \quad X = \sin\left(\frac{n\pi}{8} x\right)$ ✓

$$\begin{cases} T' + \frac{1}{4}\lambda T = 0 \\ T \neq 0 \end{cases}$$

let's say const. = 1

$T = \frac{\text{const.}}{IF} = \frac{\text{const}}{e^{\frac{1}{4}\lambda t}} = e^{-\frac{1}{4}\lambda t}$

$T = e^{-\frac{1}{4}\left(\frac{n\pi}{8}\right)^2 t}$

Superposition:

$$\underline{u_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{8} x\right) e^{-\frac{1}{4}\left(\frac{n\pi}{8}\right)^2 t}}$$

$u(x,0) = \sum_{n=1}^{100} (-1)^n \sin\left(\frac{2n\pi}{8} x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{8}x\right) e^0$

$$b_n = \frac{\int_0^8 \left(\sum_{n=1}^{100} (-1)^n \sin\left(\frac{2n\pi}{8} x\right)\right) \sin\left(\frac{n\pi}{8} x\right) dx}{\int_0^8 \sin^2\left(\frac{n\pi}{8} x\right) dx} = \frac{1}{4} \int_0^8 \sum_{n=1}^{100} (-1)^n \sin\left(\frac{2n\pi}{8} x\right) \sin\left(\frac{n\pi}{8} x\right)$$

1 b) (cont.)

$$u(x,0) = \sum_{n=1}^{100} (-1)^n \sin\left(\underset{n=2}{2}\,\frac{n\pi}{8}x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{8}x\right) e^0$$

so

$$b_2 = (-1)^n \qquad n = 1, 2, 3 \ldots$$

$$u_2(x,t) = \sum_{n=1}^{100} (-1)^n \sin\left(\frac{2n\pi}{8}x\right) e^{-\frac{1}{4}\left(\frac{2n\pi}{8}\right)^2 t}$$

1 c) $u(x,t) = u_1(x,\cancel{t}) + u_2(x,t)$, $f(x) = u_1(x) + u_2(x,0)$

Check:

$$u_{1t} + u_{2t} = \frac{1}{4}(u_{1xx} + u_{2xx})$$

- $u_t = \frac{1}{4} u_{xx} \Rightarrow 0 - 0 + \frac{1}{4} u_{xx} = \frac{1}{4} u_{xx}$ ✓

- $u(0,t) = 20 \Rightarrow u_1(0) + u_2(0,t)$

  $20 - 5(0) + 0 = 20$ ✓

- $u(8,t) = -20 \Rightarrow u_1(8) + u_2(8,t)$

  $20 - 5(8) + 0 = -20$ ✓

- $u(x,0) = f(x) = u_1(x) + u_2(x,0)$

  $20 - 5x + \sum_{n=1}^{\infty} (-1)^n 5\sin\left(\frac{2n\pi}{8} x\right) e^0 = u_1(x) + u_2(x,0)$ ✓

Name. ______________________

2. (Chapter H3. Fourier Series)

96

A- **2(a)** [20%] The function $\sin(x)$ has a Fourier cosine series defined on $0 \le x \le \pi$. The series satisfies

$$\sin(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

-1

Compute the numerical values of a_0, a_1. Then use the trig identity

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))$$

to compute the terms with $b_n = 0$.

A **2(b)** [40%] Let $g(x)$ be the Fourier cosine series for the period 2 even extension of the function $g_0(x) = 1$ on $0 \le x \le 1/2$, $g_0(x) = -1$ on $1/2 < x \le 1$. Complete the following.

(1) Graph of g_0 with its even extension on $|x| < 1$.
(2) Graph of $g(x)$ over 3 periods.
(3) Fourier cosine series coefficient formulas.
(4) Numerical values for the coefficients.
(5) Gibb's overshoot graphic on $|x| < 1$.

A **2(c)** [20%] Define $h_0(x) = \begin{cases} x & 0 \le x < \pi, \\ 2\pi - x & \pi \le x \le 2\pi, \end{cases}$ and let $h(x)$ be the 4π odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $h(-5.5\pi) + 2h(4.5\pi) + 3h(7\pi)$.

A- **2(d)** [10%] Compute a smallest period, for those functions which are periodic.

-1

(1) $f_1(x) = 3\sin(\sqrt{2}x) + 7\cos(\sqrt{2}x)$, $P = \frac{2\pi}{\sqrt{2}}$
(2) $f_2(x) = \sin(\pi x) + \sin(\pi x/7) + \sin(2\pi x/15)$, $P = \frac{15}{2}$ $P = 210$
(3) $f_3(x) = \sin(\sqrt{2}x)\cos(\sqrt{5}x)$, Not periodic ($\sqrt{2}$ & $\sqrt{5}$ irrational & do not relate)
(4) $f_4(x) = \dfrac{1+\sin(\sqrt{2}x)}{2+\cos(\sqrt{8}x)}$, $P = \frac{2\pi}{\sqrt{2}}$
(5) $f_5(x) = \displaystyle\sum_{n=1}^{10} \frac{\sin(n\pi x)}{1+n^2}$, $P = 2$

B **2(e)** [10%] A complex Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{-inx}$ of a real function $f(x)$ has to equal its Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

-2

Briefly explain how to find the equations relating the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=-\infty}^{\infty}$, but don't supply derivation details or display the equations.

Use this page to start your solution. Attach any extra pages, then re-staple.

Problem 2:

2a) $\sin(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ $\qquad 0 \le x \le \pi$

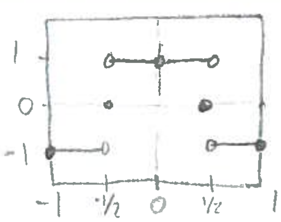
$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_{x=0}^{x=\pi} = \frac{2}{\pi} \quad , \quad a_1 = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin(x)\cos(x)}_{\frac{1}{2}(\sin(2x) - \cancel{\sin(0)}^{0})}\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\sin(2x) = \frac{1}{\pi}\left[\frac{-1}{2}\cos(2x)\right]_{x=0}^{x=\pi}$$

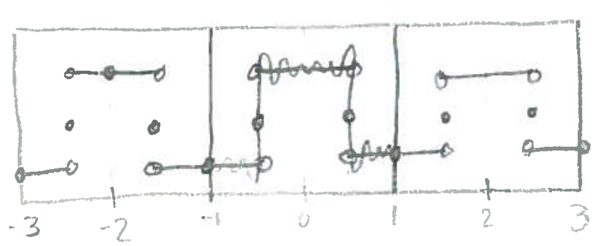
2b) Let $g(x)$ be the Fourier cosine series for the 2 even extension of the function

$$\begin{cases} g_0(x) = 1 & 0 \le x \le 1/2 \\ g_0(x) = -1 & 1/2 < x \le 1 \end{cases}$$

(1) Graph g_0 with its even extension on $|x|<1$:



(2) Graph $g(x)$ over 3 periods



2. (3) Fourier Cosine Series Coefficient formulas:

$$a_0 = \int_0^1 g(x)\,dx$$

$$a_n = 2\int_0^1 g(x)\cos(n\pi x)\,dx$$

(4) Numerical values for the coefficients:

$$a_0 = \int_0^1 \begin{cases} 1 & 0\le x\le 1/2 \\ -1 & 1/2<x\le 1 \end{cases} dx = \int_0^{1/2} 1\,dx + \int_{1/2}^1 -1\,dx = \frac{1}{2} - \frac{1}{2} = 0 = a_0.$$

$$a_n = 2\int_0^1 \begin{cases} 1 & 0\le x\le 1/2 \\ -1 & 1/2<x\le 1 \end{cases} \cos(n\pi x)\,dx = 2\int_0^{1/2}\cos(n\pi x)\,dx - 2\int_{1/2}^1 \cos(n\pi x)\,dx$$

$$a_n = 2\left[\frac{1}{n\pi}\sin(n\pi x)\right]_{x=0}^{x=1/2} - 2\left[\frac{1}{n\pi}\sin(n\pi x)\right]_{x=1/2}^{x=1}$$

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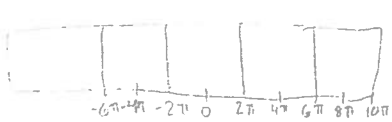
(5) Gibb's overshoot on $|x|<1$ (see (2))

2 c) $h_0(x) = \begin{cases} x, & 0\le x<\pi \\ 2\pi - x, & \pi\le x\le 2\pi \end{cases}$ let $h(x)$ be the 4π odd periodic extension of $h_0(x)$

Compute $h(-5.5\pi) + 2h(4.5\pi) + 3h(7\pi) = h(-5.5\pi + 4\pi) + 2h(4.5\pi - 4\pi) + 3h(7\pi - 4\pi - 4\pi)$



$$= h(-1.5\pi) + 2h(0.5\pi) + 3h(-\pi)$$

$$= -h(1.5\pi) + 2h(0.5\pi) - 3h(\pi)$$

$$= -(2\pi - 1.5\pi) + 2(0.5\pi) - 3(\pi)$$

$$= -\frac{\pi}{2} + \pi - 3\pi = \underline{-\frac{5}{2}\pi}$$

2 e) A complex Fourier series $\sum_{n=-\infty}^{\infty} C_n e^{-inx}$ of a real function $f(x)$ has to equal its Fourier series: $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{C_n\}_{n=-\infty}^{\infty}$.

To find the Fourier coefficients you use the law of orthogonality.

In order to relate C_n to a_n & b_n, set the complex Fourier series equal to the Fourier series (because $f(x)$ is a real function) and using initial conditions solve for C_n. X Euler's formula, not $\perp$

3. (CH H4. Finite String: Fourier Series Solution)

A

3(a) [50%] Complete the following for the finite string problem

100

$$\begin{cases} u_{tt}(x,t) &= c^2 u_{xx}(x,t), & 0 < x < L, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(L,t) &= 0, & & t > 0, \\ u(x,0) &= f(x), & 0 < x < L, & \\ u_t(x,0) &= g(x), & 0 < x < L. & \end{cases}$$

(1) Separation of variables details.

(2) Product solution boundary value problems.

(3) The product solutions (normal modes).

(4) Superposition.

(5) Series solution $u(x,t)$.

A **3(b)** [25%] Display explicit formulas for the generalized Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

A **3(c)** [25%] Evaluate the coefficients when $c = 1/3$, $L = 5$, $f(x) = 0$ and $g(x) = 50$.

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Problem 3: Finite String

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L \ , \ t > 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) & \text{''} \\ u_t(x,0) = g(x) & \text{''} \end{cases}$$

3(a)

$u = X T$

$X T'' = c^2 X'' T$

$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases} \qquad \begin{cases} T'' + c^2 \lambda T = 0 \\ T \neq 0 \end{cases} \qquad T(t) = \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda} c t) + b_n \sin(\sqrt{\lambda} c t)$$

Normal Modes $= \sin\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{n\pi c t}{L}\right)$

$\sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{n\pi c t}{L}\right)$

$\lambda > 0$

$X = \sin(\sqrt{\lambda} x)$

$0 = \sin(\sqrt{\lambda} L)$

$n\pi = \sqrt{\lambda} L$

$\frac{n\pi}{L} = \sqrt{\lambda}$

$X(x) = \sin\left(\frac{n\pi x}{L}\right)$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{n\pi c t}{L}\right)}$$

3(b) $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\int_0^L f(x)\cdot\sin\left(\frac{m\pi x}{L}\right)dx = a_n\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= a_n \underbrace{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx}_{L/2}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{m\pi x}{L}\right)dx}$$

$$U_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi c}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L g(x)\cdot\sin\left(\frac{m\pi x}{L}\right)dx = \left(\frac{n\pi c}{L}\right)b_n \underbrace{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx}_{L/2}$$

$$\boxed{\left(\frac{n\pi c}{L}\right)b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$

3(c) $c = \frac{1}{3}$, $L = 5$, $f(x) = 0$, $g(x) = 50$

$\boxed{a_n = 0}$ $f(x) = 0$ in integrand

$$\left(\frac{m\pi\left(\frac{1}{3}\right)}{5}\right)b_n = \frac{2}{5}\int_0^5 50\cdot\sin\left(\frac{m\pi x}{5}\right)dx$$

$$\left(\frac{m\pi}{15}\right)b_n = 20\left[\frac{-\cos\left(\frac{m\pi x}{5}\right)}{\frac{m\pi}{5}}\right]_0^5$$

$$\left(\frac{m\pi}{15}\right)b_n = 20\cdot\frac{5}{m\pi}\left[-\cos(m\pi) - (-\cos(0))\right]$$

$$\left(\frac{m\pi}{15}\right)b_n = \frac{100}{m\pi}\left[1 - \cos(m\pi)\right]$$

$$\boxed{b_n = \frac{1500}{(m\pi)^2}\left[1 - \cos(m\pi)\right]}$$

4. (CH H4. Rectangular Membrane) 100

Complete the following for the general membrane problem

A

$$\begin{cases} u_{tt}(x,y,t) &=& c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a,\quad 0<y<b,\quad t>0, \\ u(x,y,t) &=& 0 & \text{on the boundary}, \\ u(x,y,0) &=& f(x,y), & 0<x<a,\quad 0<y<b, \\ u_t(x,y,0) &=& g(x,y), & 0<x<a,\quad 0<y<b. \end{cases}$$

under the assumptions $a=2$, $b=4$, $c=1$, $f(x,y)=0$, $g(x,y)=100$.

(1) Separation of variables.
(2) Product solution boundary value problems.
(3) Product solutions (the normal modes).
(4) Superposition for $u(x,y,t)$.
(5) Generalized Fourier coefficient formulas.
(6) Explicit numerical values for the coefficients.

Use this page to start your solution. Attach any extra pages, then re-staple.

Problem 4 : Rectangular Membrane

$$\begin{cases} u_{tt}(x,y,t) = c^2(u_{xx} + u_{yy}) & 0<x<a,\ 0<y<b,\ t>0 \\ u(x,y,t) = 0 & \text{on boundary} \\ u(x,y,0) = f(x,y) & \text{"} \quad \text{"} \\ u_t(x,y,0) = g(x,y) & \text{"} \quad \text{"} \end{cases}$$

$a=2$, $b=4$, $c=1$, $f(x,y)=0$, $g(x,y)=100$

$$U = X(x)\,Y(y)\,T(t)$$

$$XYT'' = c^2(X''YT + XY''T)$$

$$\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

$$\begin{cases} X'' + \mu X = 0 \\ X(0) = 0 \\ X(2) = 0 \end{cases}$$

$\mu > 0$

$$X = \sin(\sqrt{\mu}\,x)$$

$$0 = \sin(\sqrt{\mu}(2))$$

$$n\pi = 2\sqrt{\mu}$$

$$\frac{n\pi}{2} = \sqrt{\mu}$$

$$X = \sin\left(\frac{n\pi x}{2}\right)$$

$$\frac{Y''}{Y} = \mu - \lambda = -(\lambda - \mu)$$

$$\begin{cases} Y'' + (\lambda - \mu)Y = 0 \\ Y(0) = 0 \\ Y(4) = 0 \end{cases}$$

$\lambda - \mu > 0$

$$Y = \sin(\sqrt{\lambda-\mu}\,y)$$

$$0 = \sin(\sqrt{\lambda-\mu}\,(4))$$

$$m\pi = 4\sqrt{\lambda-\mu}$$

$$\frac{m\pi}{4} = \sqrt{\lambda-\mu}$$

$$Y = \sin\left(\frac{m\pi y}{4}\right)$$

$$\lambda_{mn} = \left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{4}\right)^2$$

$$T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}\cos(\sqrt{\lambda_{mn}}\,t) + b_{mn}\cdot\sin(\sqrt{\lambda_{mn}}\,t)$$

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Normal Modes: $\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)\cdot\cos\left(\sqrt{\lambda_{mn}}\,t\right)$

$\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)\sin\left(\sqrt{\lambda_{mn}}\,t\right)$, $\sqrt{\lambda_{mn}} = \sqrt{\left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{4}\right)^2}$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{4}\right)\cos\left(\sqrt{\lambda_{mn}}\,t\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{4}\right)\sin\left(\sqrt{\lambda_{mn}}\,t\right)$$

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{4}\right)$$

$$a_{mn}\underbrace{\int_0^4\int_0^2 \sin^2\left(\frac{n\pi x}{2}\right)\cdot\sin^2\left(\frac{m\pi y}{4}\right)dx\,dy}_{\frac{L_x}{2}\cdot\frac{L_y}{2}} = \int_0^4\int_0^2 f(x,y)\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)dx\,y$$

$$a_{mn} = \frac{4}{L_x L_y}\int_0^4\int_0^2 f(x,y)\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)dx\,dy \quad \text{ok}$$

$$u_t(x,y,0) = g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n\left(\sqrt{\lambda_{mn}}\right)\cdot\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)$$

$$\left(\sqrt{\lambda_{mn}}\right)b_n\underbrace{\int_0^4\int_0^2 \sin^2\left(\frac{n\pi x}{2}\right)\cdot\sin^2\left(\frac{m\pi y}{4}\right)dx\,dy}_{\frac{L_x}{2}\cdot\frac{L_y}{2}} = \int_0^4\int_0^2 g(x,y)\cdot\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)dx\,dy$$

$$\left(\sqrt{\lambda_{mn}}\right)b_n = \frac{4}{L_x L_y}\int_0^4\int_0^2 g(x,y)\sin\left(\frac{n\pi x}{2}\right)\cdot\sin\left(\frac{m\pi y}{4}\right)dx\,dy$$

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.4(6) $\boxed{a_{mn} = 0}$ $f(x,y) = 0$ in integrand

$$\sqrt{\lambda_{mn}} \cdot b_n = \frac{4}{(2)(4)} \int_0^4 \int_0^2 100 \cdot \sin\left(\frac{n\pi x}{2}\right) \cdot \sin\left(\frac{m\pi y}{4}\right) dx\, dy$$

$$= \frac{1}{2} \cdot 100 \left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2 \left[\frac{-\cos\left(\frac{m\pi y}{4}\right)}{\frac{m\pi}{4}}\right]_0^4$$

$$b_n = \frac{50}{\sqrt{\lambda_{mn}}} \left[\frac{1-\cos(n\pi)}{n\pi/2}\right] \left[\frac{1-\cos(m\pi)}{m\pi/4}\right], \quad \sqrt{\lambda_{mn}} = \sqrt{\left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{4}\right)^2}$$

5. **(CH H10. Fourier Transform, Heat Equation and Gauss' Heat Kernel)**

A-

-0

99

5(a) [20%] Haberman's Fourier transform pair is

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{iwx}dx, \quad f(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx}dw.$$

Assume $f(x) = 2\pi e^{x}$ on $0 \le x \le 1$ and $f(x) = 0$ otherwise. Compute the Fourier transform $F(w)$ of $f(x)$.

Graded Details: (1) Transform integration setup, (2) Integration details, (3) Answer.

A **5(b)** [10%] Compute the magnitude of the Fourier transform $F(w) = \dfrac{e^{2iw}}{1+iw}$.

-1

B+ **5(c)** [10%] The phase θ of a Fourier transform $F(w)$ is defined as the solution of the equation $e^{i\theta} = F(w)/|F(w)|$. What is the phase of $F(w) = \frac{-1}{1+w^2}$?

A **5(d)** [50%] Gauss' heat kernel $g(x)$ (k, t constants) and the error function $\mathbf{erf}(x)$ are defined by the equations

$$g(x) = \sqrt{\frac{\pi}{kt}}\,e^{-\frac{x^2}{4kt}}, \quad \mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz.$$

The formula $FT[g(x)] = G(w) = e^{-ktw^2}$ is assumed. Solve the infinite rod heat conduction problem

$$\begin{cases} u_t(x,t) = \frac{1}{4}u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ f(x) = \begin{cases} 20 & 0 < x < 1, \\ 0 & \text{otherwise} \end{cases} & \end{cases}$$

Graded Details: (1) Fourier Transform method, (2) Convolution, (3) Heat kernel use, (4) Error function methods, (5) Final answer, expressed in terms of the error function.

A **5(e)** [10%] Compute the limits $u(0, 0+)$ and $u(1, 0+)$, which are the right hand limits of temperature $u(x,t)$ at $t = 0$ for rod locations $x = 0$ and $x = 1$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Problem: 5

a. $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$

$= \frac{1}{2\pi}\int_0^1 2\pi e^x e^{i\omega x}\, dx = \frac{2\pi}{2\pi}\int_0^1 e^{x+i\omega x}\, dx = \int_0^1 e^{(1+i\omega)x}\, dx$

$= \left[\frac{1}{1+i\omega} e^{(1+i\omega)x}\right]_0^1 = \frac{e^{(1+i\omega)} - 1}{1+i\omega}$

b. $F(\omega) = \frac{e^{2i\omega}}{1+i\omega}\,\frac{(1-i\omega)}{(1-i\omega)} = \frac{(1-i\omega)e^{2i\omega}}{1+\omega^2} = \frac{1}{1+\omega^2} - \frac{i\omega e^{2i\omega}}{1+\omega^2}$

$|F(\omega)| = \sqrt{(Re)^2 + (Im)^2} = \sqrt{\left(\frac{1}{1+\omega^2}\right)^2 + \left(\frac{\omega}{1+\omega^2}\right)^2} = \frac{1}{\sqrt{1+\omega^2}}$ ✓

c. $F(\omega) = \frac{-1}{1+\omega^2}$

$|F(\omega)| = \sqrt{\left(\frac{-1}{1+\omega^2}\right)^2 + 0} = -\frac{1}{1+\omega^2}$

$e^{i\theta} = -1$

$\theta = 0$ (Real function, phase is zero) would be correct

d. $k = \frac{1}{4}$, $FT[u] = U$:

$u_t = k u_{xx}$

$FT[u_t] = FT[k u_{xx}]$

$FT[u_t] = \frac{\partial}{\partial t} FT[u]$

$= \frac{dU}{dt}$

$FT[k u_{xx}] = k\, FT[u_{xx}]$

$= k(-i\omega)^2 FT[u]$

$= -k\omega^2 U$

$\frac{dU}{dt} = -k\omega^2 U$ (1st order ODE)

$U = U_0 e^{-k\omega^2 t}$

If $FT[f(x)] = F(\omega)$;

$U|_{t=0} = F(\omega)$ (FT of IVP taken)

$F(\omega) = U_0 e^0 = U_0$

$U = F(\omega) e^{-k\omega^2 t} = F(\omega) G(\omega)$

$IFT[U] = IFT[F(\omega) G(\omega)]$

$u(x,t) = f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv$

cont. on back →

1. $g(x-v) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-v)^2}{4kt}}$ $\qquad z^2 = \frac{(x-v)^2}{4kt}$, $z = \frac{x-v}{\sqrt{4kt}}$, $dz = \frac{dv}{\sqrt{4kt}}$

$$u(x,t) = \frac{1}{2\pi}\int_{v_1}^{v_2} 20\, e^{-z^2}\sqrt{\frac{\pi}{kt}}\sqrt{4kt}\, dz$$

(Sub in z values in order to use heat kernel)

$v_1 = \frac{-x}{\sqrt{4kt}}$, $v_2 = \frac{1-x}{\sqrt{4kt}}$

$$u(x,t) = \frac{1}{2\pi}\int_{v_1}^{v_2} 20\, e^{-z^2}\sqrt{\pi}\sqrt{4}\,\frac{\cancel{\sqrt{kt}}}{\cancel{\sqrt{kt}}}\, dz$$

$$= \frac{1}{\sqrt{\pi}}\int_{v_1}^{v_2} 20\, e^{-z^2}\, dz = \frac{20}{\sqrt{\pi}}\int_{v_1}^{v_2} 20\, e^{-z^2}\, dz$$

(Done w/ properties of integrals)

$$= 10\left(\frac{2}{\sqrt{\pi}}\right)\int_{v_1}^{v_2} e^{-z^2}\, dz = 10\left(\frac{2}{\sqrt{\pi}}\right)\left(\int_0^{v_2} e^{-z^2}\, dz - \int_0^{v_1} e^{-z^2}\, dz\right)$$

$$= 10\,\mathrm{erf}(v_2) - 10\,\mathrm{erf}(v_1)$$

2. $v_1\big|_0 = 0$, $v_1\big|_1 = \frac{-1}{\sqrt{4kt}}$

$v_2\big|_0 = \frac{1}{\sqrt{4kt}}$, $v_2\big|_1 = 0$

$$\lim_{t\to 0^+} u(0,t) = 10\,\mathrm{erf}(\infty) - 10\,\mathrm{erf}(0)$$

$$= 10$$

$$\lim_{t\to 0^+} u(1,t) = 10\,\mathrm{erf}(0) - 10\,\mathrm{erf}(-\infty)$$

$$= 10$$