Name. KEY 3150 Sample Exam 2 S2013

Partial Differential Equations 3150

Sample Midterm Exam 2
Exam Date: Monday, 22 April 2013

Instructions: This exam is timed for 50 minutes. You will be given double time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1a. (CH3. Finite String: Fourier Series Solution)

(a) [75%] Display the series formula without derivation details for the finite string problem

$$\begin{cases} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad\quad t > 0, \\ u(L,t) &= 0, \quad\quad t > 0, \\ u(x,0) &= 0, \quad 0 < x < L, \\ u_t(x,0) &= g(x), \quad 0 < x < L. \end{cases}$$

(b) [25%] Display an explicit formula for the Fourier coefficients which contains the symbols L, $g(x)$.

(a) The normal modes are $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$, $\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$ but $u(x,0)=0$, which implies no cosine terms. Then

$$u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

(b) Find $\frac{d}{dt}$ of This equation, substitute $t=0$, Then

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n^* \sin\left(\frac{n\pi x}{L}\right)\cos(0)$$

The right side is a sine series. Use orthogonality of The sine terms to obtain

$$\frac{n\pi c}{L} b_n^* = \int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \Big/ \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$b_n^* = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \Big/ \left(\frac{n\pi c}{L}\right)$$

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Name. KEY 3150 Sample Exam 2 S2013

1b. (CH3. d'Alembert's Solution: Finite String)

Let $f(x) = \begin{cases} 0.3x & 0 \le x \le 0/5, \\ 0.3(1-x) & 0.5 < x \le 1. \end{cases}$ and define $g(x) = 0$ on $0 \le x \le 1$. Assumed is d'Alembert's solution to the vibrating string problem on $0 < x < 1$, $t > 0$, which is the formula

$$u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)ds.$$ ← error

Assume $L = 1$ and $c = 1/\pi$.

(a) [25%] Define f^*, g^* as appropriate periodic extensions of f and g, respectively.

(b) [75%] Display a piecewise-defined formula for $u(x, \pi/3)$ on $0 < x < 1$.

(a) f^*, g^* are the odd 2-periodic extensions of f, g, resp.

DEF. $f^*(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ -f(-x) & -1 \le x \le 0 \end{cases}$

$f^*(x+2) = f^*(x)$

DEF: g^* defined similarly, odd and 2-periodic. Then $g^* = 0$.

(b) $u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)ds$

$= \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right)$ because $g = 0$.

$u(x, \frac{\pi}{3}) = \frac{1}{2}\left(f^*(x - \frac{c\pi}{3}) + f^*(x + \frac{c\pi}{3})\right)$

$= \frac{1}{2}\left(f^*(x - \frac{1}{3}) + f^*(x + \frac{1}{3})\right)$ because $c = 1/\pi$

For $0 < x < 1$, then $-\frac{1}{3} < x - \frac{1}{3} < \frac{2}{3}$ and $\frac{1}{3} < x + \frac{1}{3} < 1 + \frac{1}{3}$

This is perhaps the best formula, without further attempts to simplify.

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Name. KEY 3150 Sample Exam 2 S2013

2. (CH3. Heat Conduction in a Bar)

Consider the heat conduction problem in a laterally insulated bar of length 1 with one end at zero Celsius and the other end at 100 Celsius. The initial temperature along the bar is given by function $f(x)$.

$$(3) \quad \begin{cases} u_t &= c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 100, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < 1. \end{cases}$$

(a) [25%] Find the steady-state temperature $u_1(x)$.

(b) [50%] Solve the bar problem with zero Celsius temperatures at both ends, but $f(x)$ replaced by $f(x) - u_1(x)$. Call the answer $u_2(x,t)$. The answer has Fourier coefficients in integral form, unevaluated, to save time.

(c) [25%] Explain why $u(x,t) = u_1(x) + u_2(x,t)$.

(a) we solve $0 = c^2 u''$ or $u'' = 0$, Then $u = C_1 + C_2 x$ and $u(0)=0, u(1)=100$ implies $C_1 = 0, C_2 = 100$. Then $\boxed{u_1 = 100x}$

(b) $u_2 = \sum_{n=1}^{\infty} b_n e^{-(cn\pi)^2 t} \sin(n\pi x)$

$b_n = (2)\int_0^1 (f(x) - 100x)\sin(n\pi x)\,dx$

(c) Because u_1 satisfies (1) and u_2 satisfies (2),

$$(1) \begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 100 \\ u(x,0) = 0 \end{cases} \qquad (2) \begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = f(x) - 100x \end{cases}$$

Then superposition implies $u = u_1 + u_2$ satisfies the problem stated above (see (3)).

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3. (CH4. Rectangular Membrane)
Consider the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a, \quad 0<y<b, \quad t>0, \\ u(x,y,t) &= 0 & \text{on the boundary,} \\ u(x,y,0) &= f(x,y), & 0<x<a, \quad 0<y<b, \\ u_t(x,y,0) &= g(x,y), & 0<x<a, \quad 0<y<b. \end{cases}$$

Solve the problem for $a=b=1$, $c=1/\pi$, $f(x,y)=0$, $g(x,y)=1$.
Alternate problem type: Replace $f=0, g=1$ by $f=1, g=0$.

The Solution of This problem is in ASMAR, Chapter 3.7. It is easier Than The circular membrane problem. The methods of 3.7 are used to solve The drumhead problem, ch4.

The Solution is a superposition of The normal modes, found by separation of variables, as

$$\sin(m\pi x/a)\sin(n\pi y/b)\left(B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right)$$

$$\lambda_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

Then

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(m\pi x)\sin(n\pi y)\left(B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right)$$

where $\lambda_{mn} = \sqrt{m^2+n^2}$. The coefficients are found by orthogonality on $[0,1]\times[0,1]$.

$$B_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$\lambda_{mn}B^*_{mn} = 4\int_0^1\int_0^1 g(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

4. (CH4. Steady-State Heat Conduction on a Disk)
Consider the problem

$$\begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(a,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

Solve for $u(r,\theta)$ when $a = 1$ and $f(\theta) = 100\,\textbf{pulse}(\theta, 0, \pi)$, that is, $f(\theta) = 100$ on $0 \le \theta < \pi$, $f(\theta) = 0$ on $\pi \le \theta < 2\pi$.

The problem is solved in ASMAR, Chapter 4.4.
Separation of variables gives product solutions

$$\left(\frac{r}{a}\right)^n \cos(n\theta), \quad \left(\frac{r}{a}\right)^n \sin(n\theta)$$

Then

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)\left(\frac{r}{a}\right)^n$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f \, d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

Expected in your solution: Separation of variables, explain why solution r^{-n} is ignored, solve $\Theta'' + n^2\Theta = 0$ for product solutions. Finally, explain how to get the formulas for a_0, a_n, b_n from orthogonality.

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