Chapter 10.04
Introduction to Finite Element Methods

After reading this chapter, you should be able to:

1. *Understand the basics of finite element methods using a one-dimensional problem.*

In the last fifty years, the use of approximation solution methods to solve complex problems in engineering and science has grown significantly. The widespread availability of powerful digital computers and commercial computational software based on these approximation methods with efficient solution algorithms has made them practical. In this chapter, we are introducing the student to finite methods of solving differential equations. We provide an elementary background on how finite element methods work, while using a single example to illustrate the approach, and discuss the accuracy and efficacy of the method.

The single example chosen is a classical problem of a uniformly pressurized thick-walled cylinder with an axis-symmetric response (Figure 1). This problem is chosen since it is simple enough to have an analytical solution, but complex enough such that its finite element method solution can be generalized for problems that are more complicated. We must first define the problem, and then develop the exact solution so that we may compare it with the finite element methods result.

Thick-Wall Cylinder Problem

Problem Definition

Consider a thick-walled cylinder as depicted in Figure 1, with the following material properties:

Young's modulus E,
Poisson's ratio ν
inner radius a
outer radius, b
uniform internal pressure p_i
external pressure, p_o

Find the following variables in the cylinder. Plane stress state is assumed.

radial displacement, u

radial stress, σ_r
tangential stress, σ_θ

Numerical Example Problem
For demonstrating the use of approximate solution methods in solving the problem numerically, the following data is used:

$a = 0.25$ m
$b = 0.5$ m
$p_i = 200$ MPa
$p_o = 0$
$E = 207$ GPa
$\nu = 0.3$

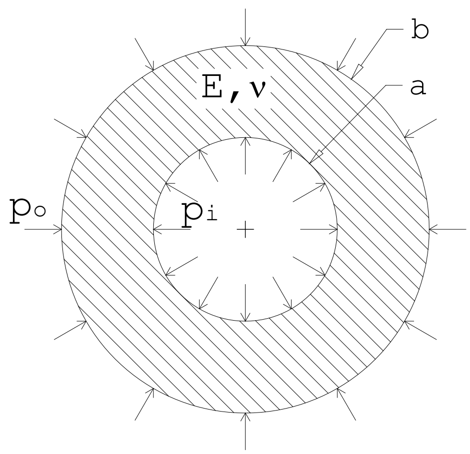

Figure 1: Pressured thick-wall cylinder problem

Mathematical Formulation

The solution of the thick-wall cylinder problem can be found by solving the equation of compatibility in polar coordinates, which is a fourth order partial differential equation of Airy stress function (1), or by using axisymmetry conditions to formulate the problem as a second order differential equation of displacement (2), or equivalent forms (potential energy, integral equation, etc.). The last approach is adopted in this paper, as it is direct and does not require inverse or semi-inverse solution methods (1, 2). The details of this approach are given in (2) and the relevant formulas are summarized as follows. The radial strain, ε_r, tangential strain, ε_θ, in terms of radial displacement, u are given as

$$\varepsilon_r = \frac{du}{dr} \tag{1}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{2}$$

The radial stress, σ_r, and tangential stress, σ_θ, in terms of radial displacement, u, are given as

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \tag{3}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\nu\frac{du}{dr} + \frac{u}{r}\right) \tag{4}$$

The governing equation for radial displacement, u, is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{5}$$

Using Equations 3-4, the boundary conditions $\sigma_r(a) = -p_i$ and $\sigma_r(b) = -p_o$ can be rewritten as

$$u'(a) + \nu\frac{u(a)}{a} = -\frac{1-\nu^2}{E}p_i \tag{6}$$

$$u'(b) + \nu\frac{u(b)}{b} = -\frac{1-\nu^2}{E}p_o \tag{7}$$

First, the exact solution is found, and then a finite element method is presented through solving the example problem. Nodal points chosen for the finite element method are uniformly spaced for convenience. Figure 2 shows how the nodal points and elements are numbered.

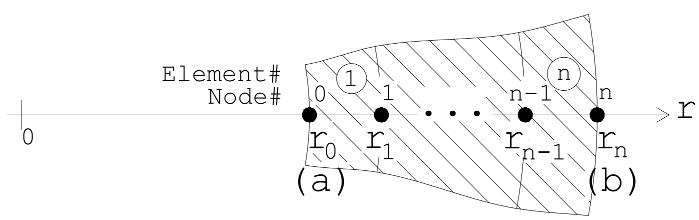

Figure 2: Numbering of nodal points and elements

Exact Solution

The exact solution of displacement can be found directly by solving the governing differential equation, Equation (5), with associated boundary conditions, Equations (6-7), and then substituting it into Equations (3-4) to give an exact solution of stresses. The exact solutions (7) of radial displacement, radial stress, and tangential stress are obtained as

$$u = \frac{1-\nu}{E}\frac{(a^2p_i - b^2p_o)r}{b^2-a^2} + \frac{1+\nu}{E}\frac{(p_i - p_o)a^2b^2}{(b^2-a^2)r} \tag{8}$$

$$\sigma_r = \frac{a^2p_i - b^2p_o}{b^2-a^2} - \frac{(p_i-p_o)a^2b^2}{(b^2-a^2)r^2} \tag{9}$$

$$\sigma_\theta = \frac{a^2p_i - b^2p_o}{b^2-a^2} + \frac{(p_i-p_o)a^2b^2}{(b^2-a^2)r^2} \tag{10}$$

Solution for Example Problem

Substituting the numerical data into Equations (8-10), the exact solution for the example problem is

$$u = \left(0.2254r + \frac{0.1047}{r}\right)\times 10^{-3} \quad (11)$$

$$\sigma_r = \left(66.67 - \frac{16.67}{r^2}\right)\times 10^{6} \quad (13)$$

$$\sigma_\theta = \left(66.67 + \frac{16.67}{r^2}\right)\times 10^{6} \quad (14)$$

Evaluating the solution at three nodal points (inner edge, $r = 0.25\,m$; mid-point, $r = 0.375\,m$; and outer edge, $r = 0.5\,m$) along the radial location for comparison, the resulted values are given in Table 1.

Table 1: Exact solution evaluated at nodal points

r (m)	0.25	0.375	0.5
u (mm)	0.4750	0.3637	0.3221
σ_r (MPa)	– 200	– 51.85	0
σ_θ (MPa)	333.3	185.2	133.3

What are Finite Element Methods?

The finite element method is a technique used to solve differential equations (ordinary or partial). They are mainly used to solve real world problems, as the differential equations that govern these problems cannot be solved exactly, or may be too intractable to be solved exactly.

The finite element methods use techniques to approximate the dependant variables of the differential equations by functions, and then reduce the unknowns in these functions to a set of simultaneous linear equations. These equations can then be solved by various numerical techniques. However, one needs to understand that finite element methods use a function, not the differential equation itself, to develop the approximate solution. This is unlike the finite difference methods, where the derivatives in the differential equations are approximated by finite divided difference methods. The functions used in the finite element methods are integral equations. In the case of the pressure vessel, these equations would model the total potential energy due to internal stresses and external loads

The Rayleigh-Ritz method can be viewed as a form of a finite element method where it reduces a continuous problem to a problem with a finite number of degrees of freedom. The Rayleigh-Ritz method is based on the principle of stationary potential energy, which states:

"Among all admissible configurations of a conservative system, those that satisfy the equations of equilibrium make the potential energy stationary with respect to small variations of displacement. If the stationary condition is a minimum, the equilibrium state is stable."

Mathematically speaking, the Rayleigh-Ritz method is a variational method, based on the idea of finding a solution that minimizes a functional. For elasticity problems, the functional is the total potential energy. The solution must be admissible, that is, satisfying internal compatibility (e.g., continuity of displacement) and essential boundary conditions. For problems where displacements are primary unknowns, essential boundary conditions are prescriptions of displacement and non-essential boundary conditions are prescriptions of stress. Since the problem considered here, the thick-walled pressured cylinder problem where the primary unknown is radial displacement, has no prescription of displacement, there is no essential boundary condition.

Potential Energy Formulation

The cylinder is assumed to be in a plane stress state which gives a strain energy density, U_0 as

$$U_0 = \frac{1}{2}\left(\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta\right) \tag{15}$$

by using Equations (1-4), we get

$$U_0 = \frac{E}{2\left(1-\nu^2\right)}\left[\left(\frac{du}{dr}\right)^2 + 2\nu\left(\frac{du}{dr}\right)\left(\frac{u}{r}\right) + \left(\frac{u}{r}\right)^2\right] \tag{16}$$

Total strain energy, U of the cylinder is

$$U = \int_{(V)} U_0 dV = \int_0^L \int_0^{2\pi} \int_a^b U_0 r dr d\theta dz = 2\pi L \int_a^b U_0 r dr \tag{17}$$

where,
$L =$ cylinder length

Work done, W by external forces (internal and external pressures) is

$$W = \int_{(S_i)} p_i u(a) ds - \int_{(S_o)} p_o u(b) ds = 2\pi a L p_i u(a) - 2\pi b L p_o u(b) \tag{18}$$

where,
$S_i =$ inner cylinder surface
$S_o =$ outer cylinder surface

The total potential energy of the cylinder, Π is found as

$$\Pi = U - W = 2\pi L\left(\int_a^b U_0 r dr - a p_i u(a) + b p_o u(b)\right) \tag{19}$$

Rayleigh-Ritz Method

The Rayleigh-Ritz method can be outlined as follows. The potential energy of the system is given as $\Pi = \Pi(u', u, r)$.
Assume a trial solution of the form: $u = f\left(r, C_0, C_1, \ldots, C_m\right)$

where $C_i's\left(i=0..m\right)$ are unknown parameters, and f is a known function. In this paper, we consider linear piecewise continuous functions.

Apply admissibility conditions to the trial solution. If there are $m-n$ admissibility conditions, we have $m-n$ equations of unknown parameters.

Solve the system of $m-n$ equations for $m-n$ unknowns $C_{n+1}\ldots C_m$, and then plug them back into the trial solution, we obtain a new trial solution that is admissible and has fewer unknowns (n unknowns) $u=f\left(r,C_0,C_1,\ldots,C_n\right)$.

Substitute the trial solution into the expression of potential energy.The stationary condition for potential energy $\delta\Pi=0$ gives

$$\left\{\frac{\partial\Pi}{\partial C_i}=0\right\}, i=0..n \tag{20}$$

Here we have a system of n algebraic equations with n unknowns. Solving this system of equations, we find the unknown parameters and thus the approximate solution for the radial displacement.

Substitute the found solution for radial displacement into Equations (3-4) to find the approximation solution for radial stress and tangential stress.

Linear Piecewise Continuous Solution for Example Problem

Consider the case of $n=2$ with uniform spacing nodal points. The step size for locating nodal points is calculated as

$$\begin{aligned} h &= (b-a)/n \\ &= (0.5-0.25)/2 \\ &= 0.125\,. \end{aligned}$$

The radial coordinates of the nodal points are $r_o=a=0.25$, $r_1=0.375$, $r_2=b=0.5$.

The displacement field is assumed to be a piecewise continuous function of two linear segments as

$$u=\begin{cases} C_0+C_1r, & 0.25\le r\le 0.375 \\ C_3+C_2r, & 0.375\le r\le 0.5 \end{cases} \tag{21}$$

To make the trial solution, Equation (21), admissible, it must be continuous at $r=0.375$, which means

$$C_0+0.375C_1=C_3+0.375C_2\text{, or} \tag{22}$$

$$C_3=C_0+0.375C_1-0.375C_2 \tag{23}$$

The trial solution, Equation (21), then becomes

$$u=\begin{cases} C_0+C_1r, & 0.25\le r\le 0.375 \\ C_0+0.375C_1-0.375C_2+C_2r, & 0.375\le r\le 0.5 \end{cases} \tag{24}$$

Substituting Equation (24) and the given numerical data into Equation (19), the total potential energy, Π in the cylinder is found as

$$\Pi = 2\pi L(78.84C_0^2 + 61.50C_0C_1 + 12.42C_0C_2 + 16.15C_1^2 + 4.659C_1C_2 + 6.912C_2^2 - 0.05000C_0 - 0.01250C_1)\times 10^9 \quad (25)$$

The condition that the total potential energy Π is stationary,

$$\left\{\frac{\partial \Pi}{\partial C_0} = 0, \frac{\partial \Pi}{\partial C_1} = 0, \frac{\partial \Pi}{\partial C_2} = 0\right\}$$

Which gives a system of algebraic equations of the unknown coefficients as

$$\begin{cases} 157.7C_0 + 61.50C_1 + 12.42C_2 = 0.05000 \\ 61.50C_0 + 32.31C_1 + 4.659C_2 = 0.01250 \\ 12.42C_0 + 4.659C_1 + 13.82C_2 = 0 \end{cases} \quad (26)$$

The unknown coefficients are found as

$$\begin{cases} C_0 = 0.0006737 \\ C_1 = -0.0008496 \\ C_2 = -0.0003191 \end{cases} \quad (27)$$

Substituting Equation (27) into Equation (24), the approximate solution for radial displacement is

$$u = \begin{cases} 0.0006737 - 0.0008496r, & 0.25 \le r \le 0.375 \\ 0.0004748 - 0.0003191r, & 0.375 \le r \le 0.5 \end{cases} \quad (28)$$

Substituting the numerical data and displacement solution from Equation (28) into Equations (3-4), we find the radial and tangential stresses as

$$\sigma_r = \begin{cases} \dfrac{45.97}{r} - 251.2, & 0.25 < r < 0.375 \\ \dfrac{32.40}{r} - 94.38, & 0.375 < r < 0.5 \end{cases} \quad (29)$$

$$\sigma_\theta = \begin{cases} \dfrac{153.2}{r} - 251.2, & 0.25 < r < 0.375 \\ \dfrac{108.0}{r} - 94.38, & 0.375 < r < 0.5 \end{cases} \quad (30)$$

The solution of the radial displacement is continuous, since we have forced the trial solution to be admissible from the beginning, while the solutions for stresses are discontinuous at the interior knot ($r = 0.375$) between the two segments (elements). To have reasonable results, in practice, the stress value at the interior knot is taken as the average of two stress values. The numerical solution with $n = 2$ of the example problem is given in Table 4.

Table 4: Numerical solution, finite element method ($n = 2$)

r (m)	0.25	0.375	0.5
u (mm)	0.4613	0.3551	0.3152
σ_r (MPa)	– 67.35	– 68.32	– 29.58
σ_θ (MPa)	361.7	175.5	121.6

The exact solution and numerical solutions with various values of number of nodal points, $n = 2$, 3, and 4, are given in Figure 5 for radial displacement, and Figure 6 for radial stress.

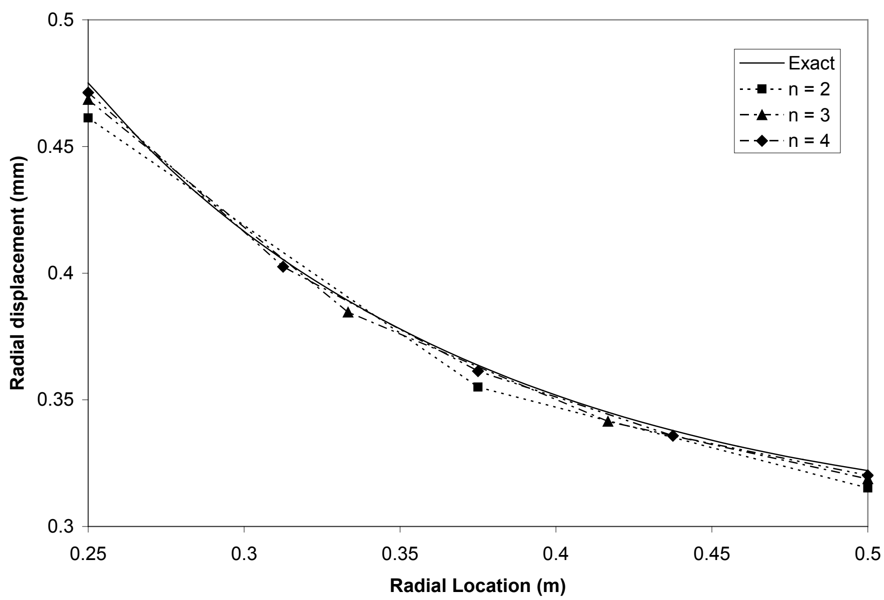

Figure 5: Radial displacement as a function of radial location (Finite element method)

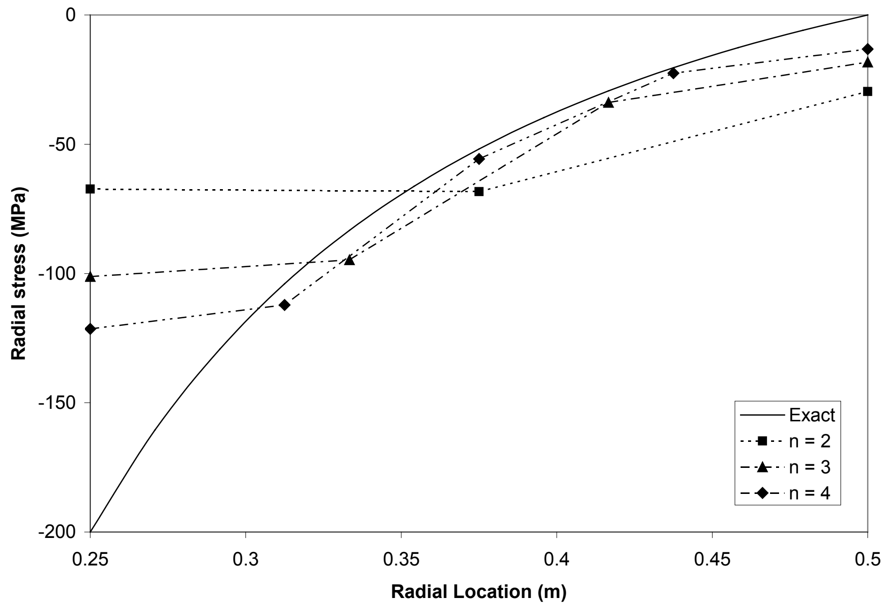

Figure 6: Solution of radial stress as a function of radial location

The solution plots show that the approximate solutions approach the exact solution as the number of piecewise continuous functions increase. However, they do not satisfy the boundary conditions of radial stress. The assumption of the piecewise continuous solution as opposed to a continuous solution makes computation easier for a high number of segments in the piecewise functions, but it has the drawback of the discontinuity of stresses at the interior knots of the piecewise continuous function.

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PARTIAL DIFFERENTIAL EQUATIONS	
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