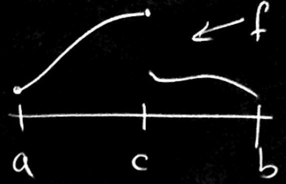
H3.4-1



Corrected Calculus: Integration by Parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du + uv\Big|_{c+}^{c-}$$

$$\int_a^c u\,dv = \lim_{h\to 0^+}\int_{a+h}^{c-h} uv'\,dx = \lim_{h\to 0^+}\left(uv\Big|_{a+h}^{c-h} - \int_{a+h}^{c-h} v\,du\right) = u(c-)v(c-) - u(a+)v(a+) - \int_a^c v\,du$$

$$\int_c^b u\,dv = \lim_{h\to 0^+}\int_{c+h}^{b-h} uv'\,dx = \lim_{h\to 0^+}\left(uv\Big|_{c+h}^{b-h} - \int_{c+h}^{b-h} v\,du\right) = u(b-)v(b-) - u(c+)v(c+) - \int_c^b v\,du$$

H3.4-2 Assume f, f' piecewise smooth and f continuous (no jumps)

Show that the Fourier Series obtained with Term-by-term differentiation has Fourier Coefficients for $f'(x)$. No claim that $f'(x) =$ this Fourier series.

Let $\lambda_n = \frac{n\pi}{L}$

$$a_n = \frac{2}{2L}\int_{-L}^{L} f(x)\cos(\lambda_n x)dx \qquad b_n = \frac{2}{2L}\int_{-L}^{L} f(x)\sin(\lambda_n x)dx$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$A_n = \frac{2}{2L}\int_{-L}^{L} f'(x)\cos(\lambda_n x)dx \qquad B_n = \frac{2}{2L}\int_{-L}^{L} f'(x)\sin(\lambda_n x)dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)dx$$

$$f(x) \sim a_0 + \sum_{1}^{\infty} a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x)$$

$$f' \sim A_0 + \sum_{1}^{\infty} A_n \cos(\lambda_n x) + B_n \sin(\lambda_n x)$$

Term-by-term diff. $= 0 + \sum_{1}^{\infty} -\lambda_n a_n \sin(\lambda_n x) + \lambda_n b_n \cos(\lambda_n x)$

careful parts $\begin{cases} u = f(x) \\ dv = \sin(\lambda_n x)dx \end{cases}$

Need Lemmas

$$\begin{cases} A_0 = 0 \\ A_n = \lambda_n b_n \\ B_n = -\lambda_n a_n \end{cases}$$

Lemma 1 $A_n = \lambda_n b_n$

f cont

Details: $A_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos(\lambda_n x)dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\lambda_n x)dx = \frac{1}{L}\left(uv\Big|_{-L}^{L} - \int_{-L}^{L} v\,du + uv\Big| ? \right)$

$$b_n = \frac{1}{L}\left(-\frac{f(x)\cos(\lambda_n x)}{\lambda_n}\Big|_{-L}^{L} - \int_{-L}^{L} -\frac{\cos(\lambda_n x)}{\lambda_n} f'(x)dx \right)$$

$$b_n = \frac{1}{\lambda_n L}\left(-f(L)\cos(\lambda_n L) + f(-L)\underbrace{\cos(-\lambda_n L)}_{\text{even}}\right)$$

$$+ \frac{1}{\lambda_n L}\int_{-L}^{L} f'(x)\cos(\lambda_n x)\,dx$$

$$= \frac{1}{\lambda_n L}\cos(\lambda_n L)\left(-f(L) + f(-L)\right)$$

$$+ \frac{1}{\lambda_n} A_n$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\lambda_n x)\,dx$$

parts $\begin{cases} u = f(x) \\ dv = \cos(\lambda_n x)\,dx \end{cases}$

$$= \frac{1}{L}\left(uv\Big|_{-L}^{L} - \int_{-L}^{L} f'(x)\,\frac{\sin(\lambda_n x)}{\lambda_n}\,dx \right)$$

$$= \frac{1}{L}\Big(f(L)\sin(\lambda_n L) - f(-L)\sin(-\lambda_n L)\Big)$$

$$-\frac{1}{\lambda_n} B_n = \boxed{-\frac{1}{\lambda_n} B_n}$$

H3.4-3

f : continuous except at $x = x_0$, $f(x_0+) = \beta$, $f(x_0-) = \alpha$

$\lambda_n = \frac{n\pi}{L}$

$\lambda_n L = n\pi$

* ① Lemma: $a_n = \frac{2}{L}(\alpha - \beta)\frac{\sin(\lambda_n x_0)}{\lambda_n} - \frac{1}{\lambda_n} B_n$

Details: $a_n = \frac{2}{L}\int_0^L f(x)\cos(\lambda_n x)\,dx = \frac{2}{L}\left(uv\Big|_0^L + uv\Big|_{x_0+}^{x_0-} - \int_0^L v\,du \right) = \frac{2}{L}\left(f(x)\frac{\sin(\lambda_n x)}{\lambda_n}\Big|_0^L + f(x)\frac{\sin(\lambda_n x)}{\lambda_n}\Big|_{x=x_0+}^{x=x_0-} - \int_0^L f'(x)\frac{\sin(\lambda_n x)}{\lambda_n}\,dx \right)$

$\begin{cases} u = f(x) \\ dv = \cos(\lambda_n x)\,dx \end{cases}$

parts

$= \frac{2}{L}\left(0 + \frac{f(x_0-)\sin(\lambda_n x_0) - f(x_0+)\sin(\lambda_n x_0)}{\lambda_n} \right) - \frac{1}{\lambda_n} B_n$

ⓑ Lemma $\quad b_n = \frac{2}{L}\left(\frac{-f(L)(-1)^n + f(0)}{\lambda_n}\right) - (-\beta+\alpha)\frac{\cos(\lambda_n x_a)}{\lambda_n}\cdot\frac{2}{L} + \frac{1}{\lambda_n}A_n$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\lambda_n x)\,dx \qquad A_n = \frac{2}{L}\int_0^L f'(x)\cos(\lambda_n x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\lambda_n x)\,dx \qquad B_n = \frac{2}{L}\int_0^L f'(x)\sin(\lambda_n x)\,dx$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad A_0 = \frac{1}{L}\int_0^L f'(x)\,dx$$

H3.4-6

$$e^x \sim a_0 + \sum_1^\infty a_n \cos(\lambda_n x)$$

$f(x) = e^x$

$\lambda_n = \frac{n\pi}{L}$, $a_n = \frac{1}{L}\int_0^L f(x)\cos(\lambda_n x)\,dx$

Term-by-term $\quad 0 + \sum_1^\infty -\lambda_n a_n \sin(\lambda_n x)$ $\quad \boxed{\text{OK}, = e^x}$ $\quad f'(x) = e^x$

No error $\quad -\lambda_n a_n = B_n = \frac{1}{L}\int_0^L f'(x)\sin(\lambda_n x)\,dx$

Term-by-term $\quad 0 + \sum -\lambda_n^2 a_n \cos(\lambda_n x)$ $\quad \boxed{\text{Not OK} \neq e^x}$

violates eq (13), H3.4

(13) says $\quad (e^x)'' = \frac{1}{L}\left(g(L) - g(0)\right) + \sum_1^\infty \left[\lambda_n B_n + \frac{2}{L}\left((-1)^n g(L) - g(0)\right)\right]\cos(\lambda_n x)$ $\quad g(x) = f'(x) = e^x$

also $e^x = a_0 + \sum_1 a_n \cos(\lambda_n x)$ by Fourier Conv. Thm

Match coeff

$$a_0 = \frac{1}{L}(e^L - 1)$$

$$a_n = \lambda_n B_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$
$$= -\lambda_n^2 a_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

and

$$a_0 = \frac{1}{L}\int_0^L \underbrace{e^x}_{f(x)} dx = \frac{1}{L}(e^L - 1)$$

$$a_n = \frac{\frac{2}{L}\left((-1)^n e^L - 1\right)}{1 + \lambda_n^2}$$

H3.4-8

$$\begin{cases} u_t = k u_{xx} \\ u_x = 0 \text{ at } x=0 \text{ \& } x=L \\ u(x,0) = f(x) \end{cases}$$

Trial Sol based on Fourier cosine series

$$u(x,t) = \sum_1^\infty b_n(t)\cos(\lambda_n x) + b_0(t)$$

$$b_0(t) = a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$b_n(t) = a_n e^{-\lambda_n^2 kt} = \frac{2}{L}\left(\int_0^L f(x)\cos(\lambda_n x)\,dx\right)e^{-\lambda_n^2 kt}$$

① Fourier Conv Thm $\Rightarrow f(x) = a_0 + \sum_1^\infty a_n \cos(\lambda_n x) = u(x,0)$

BC $u(x,0) = f(x)$ Satisfied

② $u_x = \sum_1^\infty -b_n(t)\lambda_n \sin(\lambda_n x)$

H3.4-8

$$\begin{cases} u_t = k u_{xx} \\ u_x = 0 \text{ at } x=0 \ \& \ x=L \\ u(x,0) = f(x) \end{cases}$$

Trial Sol based on Fourier cosine series

$$u(x,t) = \sum_1^\infty b_n(t)\cos(\lambda_n x) + b_0(t)$$

$$b_0(t) = a_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$b_n(t) = a_n e^{-\lambda_n^2 kt} = \frac{2}{L}\left(\int_0^L f(x)\cos(\lambda_n x)dx\right)e^{-\lambda_n^2 kt}$$

① Fourier Conv Thm $\Rightarrow f(x) = a_0 + \sum_1^\infty a_n \cos(\lambda_n x) = u(x,0)$

BC $u(x,0) = f(x)$ satisfied

② $u_x = \sum_1^\infty -b_n(t)\lambda_n \sin(\lambda_n x)$ by diff. of cosine series H3.4

Then $u_x = 0$ at $x=0$ and $x=L$

BC $u_x(0,t)=0$, $u_x(L,t)=0$ are satisfied

③ Check $u_t = k u_{xx}$

$$u_t = \sum_1^\infty b_n'(t)\cos(\lambda_n x) = \sum_1^\infty -\lambda_n k\, b_n(t)\cos(\lambda_n x)$$

<u>Last step</u>, Show $u_t = k\,u_{xx}$

$$k\,u_{xx} = k\sum_1^\infty -\lambda_n^2\, b_n(t)\cos(\lambda_n x)$$

$$u_t = \sum_1^\infty -\lambda_n^2\, k\, b_n(t)\cos(\lambda_n x) \quad \text{upper board}$$

$$= k\,u_{xx}\,!$$

$$u_{xx} = (u_x)_x = \sum_1 -\lambda_n^2\, b_n(t)\cos(\lambda_n x) \quad \text{by (13)}$$

$+$ free terms

1) $\frac{1}{L}\left(g(L) - g(0)\right)$ $\qquad g(x) = u_x(x,t)$

2) $\frac{2}{L}\left((-1)^n g(L) - g(0)\right)$ $\qquad$ So $g(L)=0$, $g(0)=0$ (Insulation problem)

all free terms $= 0$

H3.4-9

$$\begin{cases} u_t = k u_{xx} + q(x,t) \\ u_x = 0 \text{ at } x = 0 \;\&\; x = L \\ u(x,0) = f(x) \end{cases}$$

H3.4-9

$$\begin{cases} u_t = k u_{xx} + q(x,t) \\ u = 0 \text{ at } x=0 \text{ \& } x=L \\ u(x,0) = f(x) \end{cases}$$

Expect $u(x,t) = \sum_{1}^{\infty} b_n \sin(\lambda_n x)\, e^{-\lambda_n^2 k t}$ if $q \equiv 0$

Steal H3.4-8 idea

Define $u(x,t) = \sum_{1}^{\infty} B_n(t) \sin(\lambda_n x)$

$$u_t = \sum_{1}^{\infty} B_n'(t) \sin(\lambda_n x)$$

$$u_{xx} = \sum_{1}^{\infty} B_n(t)(-\lambda_n^2) \sin(\lambda_n x)$$

H3.4-9

$$\begin{cases} u_t = k u_{xx} + q(x,t) \\ u = 0 \text{ at } x=0 \,\&\, x=L \\ u(x,0) = f(x) \end{cases}$$

Expect $u(x,t) = \sum_1^\infty b_n \sin(\lambda_n x)\, e^{-\lambda_n^2 k t}$ if $q \equiv 0$

Steal H3.4-8 idea

Define $u(x,t) = \sum_1^\infty B_n(t) \sin(\lambda_n x)$

To diff sine series, need by (13) condition $f(0) = f(L) = 0$

$$u_t = \sum_1^\infty B_n'(t) \sin(\lambda_n x)$$

$$u_{xx} = \sum_1^\infty B_n(t)(-\lambda_n^2) \sin(\lambda_n x)$$

$$q(x,t) = \sum_1^\infty q_n(t) \sin(\lambda_n x)$$ by Fourier Conv. Thm

Last step

$$u_t = k u_{xx} + g(x,t)$$

$$\sum_1^\infty B_n'(t)\sin(\lambda_n x) = \sum_1^\infty -\lambda_n^2 k B_n(t)\sin(\lambda_n x) + \sum_1^\infty g_n(t)\sin(\lambda_n x)$$

Fourier series both sides, match coeff

$$\boxed{B_n'(t) = -\lambda_n^2 k B_n(t) + g_n(t)}$$