Half-Line problems

DCT [Jpeg]

DST

Discrete Versions

Fourier Cosine transform

$$f(x) = \int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega, \quad F_c(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx$$ ← Cosine pair

Fourier Sine transform

$$f(x) = \int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega, \quad F_s(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$$ ← Sine pair

Fourier Sine Convolution Thm

Used to extend Fourier's method to half-line problems

Example
$$\begin{cases} u_t = k\,u_{xx} & 0<x<\infty,\ t>0 \\ u(0,t)=0 & t>0 \\ u(x,0)=f(x) & 0<x<\infty \end{cases}$$

Method of separation of variables

$u = X(x)\,T(t)$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0)=0 \end{cases} \qquad T' + \lambda k T = 0$$

$\downarrow$ $X(x) = \sin(\sqrt{\lambda}\,x)$ $\qquad$ $\downarrow$ $T(t) = e^{-\lambda k t}$

Superposition

$u = $ sum of $\sin(\sqrt{\lambda}x)e^{-\lambda k t}$ over $0<\lambda<\infty$

Let $\omega = \sqrt{\lambda}$

$$u = \sum_\omega A(\omega)\sin(\omega x)e^{-k\omega^2 t} = \int_0^\infty A(\omega)\sin(\omega x)e^{-k\omega^2 t}\,d\omega$$

$$f(x) = u(x,0) = \int_0^\infty A(\omega)\sin(\omega x)\,d\omega$$

Theorem $A(\omega) = \frac{2}{\pi}\int_0 f(x)\sin(\omega x)\,dx$

Trick Odd extension

$$g(x) = \begin{cases} f(x) & 0<x<\infty \\ -f(-x) & -\infty<x<0 \end{cases}$$

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DST

Fourier Sine transform

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↑

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$T(t) = e^{-\lambda k t}$

Superposition

$u =$ sum of $\sin(\sqrt{\lambda}x)e^{-\lambda k t}$ over $0 < \lambda < \infty$

Let $\omega = \sqrt{\lambda}$

$$u = \sum_{\omega} A(\omega)\sin(\omega x)e^{-k\omega^2 t} = \int_0^\infty A(\omega)\sin(\omega x)e^{-k\omega^2 t}\,d\omega$$

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Theorem $A(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(x)\sin(\omega x)\,dx$

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Mon 4pm OSH 113
Tues 4pm L105, WEB

Wave

problem 1, 4 April

blackboard photos

Heat

problem 1, same ref

Fourier Transform

problem 1, same ref

Fourier Transform

problems 2,3 today

problem 2

$$\begin{cases} u_t = k u_{xx} + c u_x & -\infty < x < \infty \\ & t > 0 \\ u(x,0) = f(x) \end{cases}$$

problem 3

$$\begin{cases} u_t = \frac{1}{4} u_{xx} \\ u(x,0) = f(x) = \begin{cases} 50 & 0 < x < 1 \\ 100 & -1 < x < 0 \\ 0 & \text{else} \end{cases} \end{cases}$$

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Thm 1 $\frac{d}{dt} FT[v(x,t)] = FT[v_t(x,t)]$

Thm 2 $FT[v_x(x)] = (-i\omega) FT[v(x)]$

problem 2

$$FT[u_t(x,t_0)] = k\, FT[u_{xx}(x,t_0)] + c\, FT[u_x(x,t_0)]$$

$$\frac{d}{dt} FT[u(x,t_0)] = k(-i\omega)^2 FT[u(x,t_0)] + c(-i\omega) FT[u(x,t_0)]$$

thm 1

thm 2

Problem 2

$$FT[u_t(x,t_0)] = k\, FT[u_{xx}(x,t_0)] + c\, FT[u_x(x,t_0)]$$

$$\frac{d}{dt} FT[u(x,t_0)] = k(-i\omega)^2 FT[u(x,t_0)] + c(-i\omega) FT[u(x,t_0)]$$

thm 1

thm 2

$$\frac{d}{dt} U = -k\omega^2 U - ic\omega U \quad , \quad U = FT[u(x,t_0)]$$

Problem 2

$$FT[u_t(x,t_0)] = k\, FT[u_{xx}(x,t_0)] + c\, FT[u_x(x,t_0)]$$

thm 1

$$\frac{d}{dt} FT[u(x,t_0)] = k(-i\omega)^2 FT[u(x,t_0)] + c(-i\omega) FT[u(x,t_0)]$$

thm 2

$$\frac{d}{dt} U = -k\omega^2 U - ic\omega U \quad , \quad U = FT[u(x,t_0)]$$

$$U = U_0 e^{-k\omega^2 t - ic\omega t}$$

Set $t_0 = 0$: $FT[u(x,0)] = U_0 e^0$ or $F(\omega) = U_0$

$$U = F(\omega) e^{-k\omega^2 t - ic\omega t}$$

Problem 2

$$FT[u_t(x,t_0)] = k\, FT[u_{xx}(x,t_0)] + c\, FT[u_x(x,t_0)]$$

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thm 1 thm 2

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Set $t_0 = 0$: $FT[u(x,0)] = U_0 e^0$ or $F(\omega) = U_0$

$$U = F(\omega) e^{-k\omega^2 t - ic\omega t}$$

$x + ct$

$$u(x,t) = \int_{-\infty}^{\infty} U e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) e^{-k\omega^2 t - i(x+ct)\omega} d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega) e^{-k\omega^2 t} e^{-ix\omega} d\omega \Big|_{x = x+ct}$$

$$= V(x+ct)$$

Problem 2

$$FT[u_t(x,t_0)] = k\, FT[u_{xx}(x,t_0)] + c\, FT[u_x(x,t_0)]$$

$$\frac{d}{dt} FT[u(x,t_0)] = k(-i\omega)^2 FT[u(x,t_0)] + c(-i\omega) FT[u(x,t_0)]$$

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$$\frac{d}{dt} U = -k\omega^2 U - ic\omega U \quad , \quad U = FT[u(x,t_0)]$$

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Set $t_0 = 0$: $FT[u(x,0)] = U_0 e^0$ or $F(\omega) = U_0$

$$U = F(\omega) e^{-k\omega^2 t - ic\omega t}$$

$$u(x,t) = \int_{-\infty}^{\infty} U e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) e^{-k\omega^2 t - i(x+ct)\omega} d\omega$$

$x+ct$

$$= \int_{-\infty}^{\infty} \left(F(\omega) e^{-k\omega^2 t}\right) e^{-iX\omega} d\omega \Big|_{X = x+ct}$$

$$= y(X) = y(x+ct)$$

$$y(x) = \int_{-\infty}^{\infty} Y(w) e^{-ixw} \, dw$$

$$Y(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(x) e^{ixw} \, dx$$

$$Y(w) = F(w) e^{-kw^2 t}$$

$$y(x) = \int_{-\infty}^{\infty} Y(\omega) e^{-ix\omega} d\omega$$

$$Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(x) e^{ix\omega} dx$$

$$Y(\omega) = F(\omega) e^{-k\omega^2 t} = F(\omega) G(\omega)$$

$$G(\omega) = FT[g(x)], \quad g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$$

Problem 2

$$y(x) = \int_{-\infty}^{\infty} Y(\omega) e^{-ix\omega} d\omega$$

$$Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(x) e^{ix\omega} dx$$

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$y(x) = f * g$ by Conv thm

rest of Problem 2

$$u(x,t) = y(x+ct) = f * g$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)g(x-v)\,dv$$

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x+ct-v)^2}{4kt}}\,dv$$

problem 3

$u(x,t) =$ convolution of $f(x)$ and $g(x)$,

$g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$

$k = \frac{1}{4}$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\,g(x-v)\,dv = \frac{1}{2\pi}\int_{-1}^{0} 100\, g(x-v)\,dv + \frac{1}{2\pi}\int_{0}^{1} 50\, g(x-v)\,dv$$

Change $g(x-v)$ into constant $\times\, e^{-z^2}$

$\frac{(x-v)^2}{4kt} = z^2$, use $z = \frac{v-x}{\sqrt{4kt}}$, $dz = \frac{dv}{\sqrt{4kt}}$

$\Big\}$ Calculus to convert integrals to erf($\cdot$)

$$\frac{1}{2\pi}\int_{-1}^{0} 100\, g(x-v)\,dv = \frac{1}{2\pi}\int_{V_2}^{V_1} 100\, e^{-z^2}\cdot\sqrt{\frac{\pi}{kt}}\sqrt{4kt}\,dz$$

$$V_2 = \frac{-1-x}{\sqrt{4kt}},\quad V_1 = \frac{0-x}{\sqrt{4kt}}$$

$$= \frac{1}{\sqrt{\pi}}(100)\int_{V_2}^{V_1} e^{-z^2}\,dz$$

$$= (50)\frac{2}{\sqrt{\pi}}\left(\int_0^{V_1} - \int_0^{V_2}\right)$$

$$= 50\left(\operatorname{erf}(V_1) - \operatorname{erf}(V_2)\right)$$

Problem 3

$$\begin{cases} u_t = \frac{1}{4}u_{xx} \\ u(x,0) = f(x) = \begin{cases} 50 & 0<x<1 \\ 100 & -1<x<0 \\ 0 & \text{else} \end{cases} \end{cases}$$

answer

$$u = 25\operatorname{erf}(V_1) - 50\operatorname{erf}(V_2) + 25\operatorname{erf}(V_3)$$

$$V_1 = \frac{-x}{\sqrt{4kt}},\quad V_2 = \frac{-1-x}{\sqrt{4kt}},\quad V_3 = \frac{1-x}{\sqrt{4kt}}$$