d'Alembert's Solution

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ & t \geq 0 \\ u(x,0) = f(x), \; u_t(x,0) = g(x) & \end{cases}$$

Shape — Speed

Does not have endpoint BC

Not a Series Solution

Not easy to use on $0 < x < L$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$u(x,t) = \frac{1}{2} f(x+ct) + f(x-ct) + 0 \quad \text{when } g = 0$$

= Superposition of 2 traveling wave

Method: Change variables

$$\xi = x + ct, \quad \eta = x - ct$$

$$\longrightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} = 0 \Rightarrow u = F(\xi) + G(\eta)$$

Next find F, G. Change back for $u(x,t)$

Method: Change variables

$$\xi = x + ct, \quad \eta = x - ct$$

$$\longrightarrow \frac{\partial^2 u}{\partial\eta\,\partial\xi} = 0 \Rightarrow u = F(\xi) + G(\eta)$$

Next find F, G. Change back for $u(x,t)$

Book

Separation of variables on $0 \le x \le L$

prod Sol $u = XT$

$u =$ Superposition of prod Sols

H4.3 BC for Wave Equation $u_{tt} = c^2 u_{xx}$

Dirichlet problem $u(0,t)=0, \; u(L,t)=0$ clamped, fixed end

$u(x,0) = f(x) =$ shape , $u_t(x,0) = g(x) =$ speed

Neumann problem $u_x(0,t)=0, \; u_x(L,t)=0$ Free ends

+ shape + speed

Robin Problem $u_x(0,t) = h_1(u(L,t)-u_1)$, $u_x(L,t) = -h_2(u(L,t)-u_2)$

Mixed Problem

H4.3 BC for Wave Equation $u_{tt} = c^2 u_{xx}$

Dirichlet problem $\quad u(0,t)=0,\ u(L,t)=0 \quad$ clamped, fixed end

$u(x,0)=f(x)=$ shape, $u_t(x,0)=g(x)=$ speed

Neumann problem $\quad u_x(0,t)=0,\ u_x(L,t)=0 \quad$ Free ends

+ shape + speed

Robin Problem $\quad u_x(0,t)= h_1(u(L,t)-u_1),\ u_x(L,t)=-h_2(u(L,t)-u_2)$ ∓ Newton Cooling Law

+ shape + speed

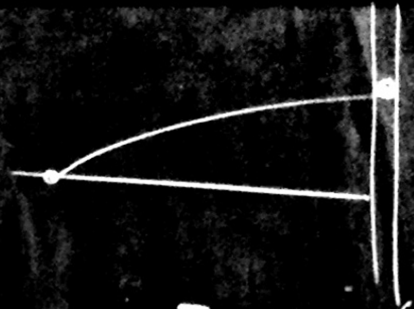
Mixed Problem

Ex. $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t)=0, \quad u_x(L,t)=0 \\ + \text{Shape} + \text{speed} \end{cases}$

Non-Homog problem

$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t)=0, \quad u(L,t)=f_1(t) \\ + \text{Shape} + \text{speed} \end{cases}$

Ex. $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0, \quad u_x(L,t) = 0 \\ + \text{shape} + \text{speed} \end{cases}$



Non-Homog problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0, \quad u(L,t) = f_1(t) \\ + \text{shape} + \text{speed} \end{cases}$$

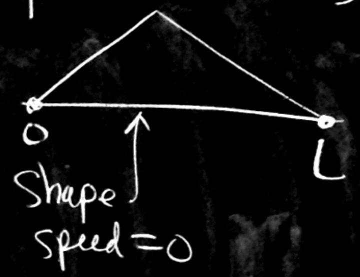
H4.4 Vibrations of a String, Fixed Ends

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u = 0 \text{ at } x = 0 \text{ and } x = L \quad \text{clamped} \\ u(x,0) = f(x), \; u_x(0,t) = g(x) \end{cases}$$

Prod Sols $= X\,T$

$u_n = X_n(x)\, T_n(t)$

Plucked String



$u =$ Superposition of prod sols

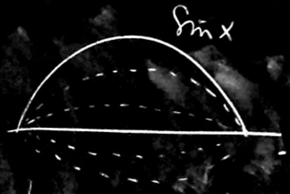
$$u = \sum_{1}^{\infty} X_n T_n$$

$$\left\{ X'' + \lambda X = 0 \right.$$

Visualize prod sol XT

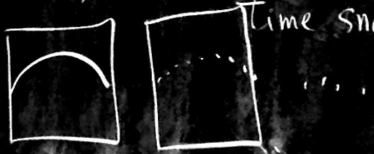
$$\overbrace{\sin(x)}^{X}\ \overbrace{\sin(10t)}^{T}$$

$$= \sin(10t)\sin(x)$$



Movie frame

$t \rightarrow$



Separate vars X, T:

$$X T'' = c^2 X'' T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

u = Superposition of prod sols

$$u = \sum_{1}^{\infty} X_n T_n$$

$$\begin{cases} X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0 \\ T'' + \lambda c^2 T = 0, \quad T \neq 0 \end{cases}$$

answers

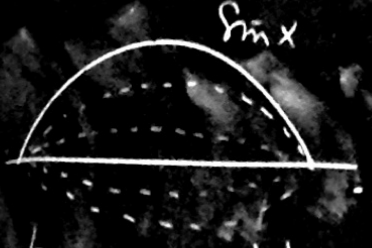
$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{aligned} T_n &= c_1 \cos\left(\frac{n\pi c t}{L}\right) + c_2 \sin\left(\frac{n\pi c t}{L}\right) \\ &= c_3 \sin\left(\frac{n\pi c t}{L} + \phi\right) \\ &= c_n \sin\left(\frac{n\pi c t}{L} + \phi_n\right) \end{aligned}$$

Visualize prod sol XT

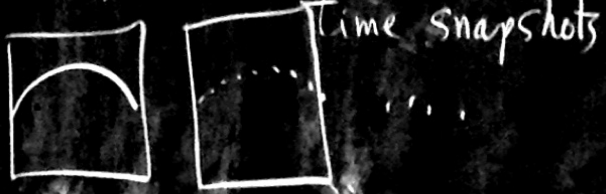
$$\underbrace{\sin(x)}_{X} \; \underbrace{\sin(10t)}_{T}$$

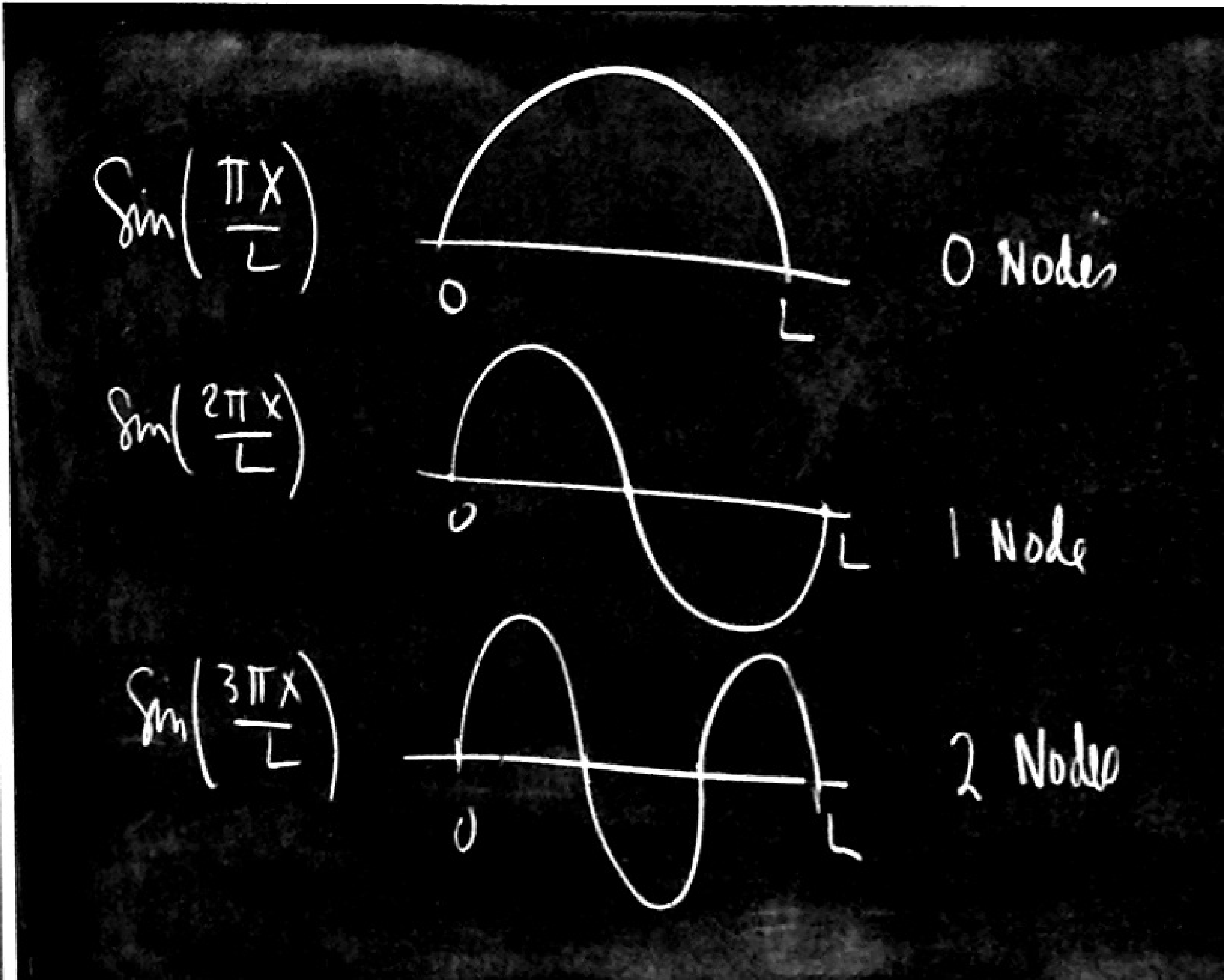
$$= \sin(10t)\sin(x)$$



Movie frame

$t \rightarrow$




Sin(πX/L)
0
L
0 Nodes
Sin(2πX/L)
0
L
1 Node
Sin(3πX/L)
0
L
2 Nodes

H4.4 Vibrations of a String, Fixed Ends

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u = 0 \text{ at } x=0 \text{ and } x=L \quad \text{clamped} \\ u(x,0) = f(x), \ u_x(0,t) = 0 \end{cases}$$

$$u = \sum_1^\infty X_n(x)\left(c_n \cos\left(\frac{n\pi ct}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

$$u_x = \sum_1^\infty X_n'(x)(\text{ same }) \longrightarrow 0 = \sum \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)(\underset{t \to 0}{\text{same}}) = \text{Fourier Series}$$

H4.4 Vibrations of a String, Fixed Ends

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u = 0 \text{ at } x=0 \text{ and } x=L \quad \text{clamped} \\ u(x,0) = f(x), \; u_t(x,0) = 0 \end{cases}$$

$$u = \sum_1^\infty X_n(x)\left(c_n \cos\left(\frac{n\pi ct}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

$$u_x = \sum_1^\infty X_n'(x)(\text{same})$$

$t=0$

$$g(x) = \sum_1^\infty X_n \left(\frac{d}{dt}\text{same}\right) = \text{Fourier series}$$

$$f(x) = \sum_1 X_n(x)\, c_n = \text{Fourier sine series} \qquad g(x) = \sum \frac{n\pi}{L}$$

c_n = Fourier coeff.

H4.4 Vibrations of a String, Fixed Ends

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u=0 \text{ at } x=0 \text{ and } x=L \quad \text{clamped} \\ u(x,0)=f(x), \; u_t(x,0)=0 \end{cases}$$

$$u = \sum_{1}^{\infty} X_n(x)\left(c_n \cos\left(\frac{n\pi ct}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$