Fourier Series, problem!

$A \sin(\omega x - \alpha)$

Amplitude $= A$

period $= \frac{2\pi}{\omega}$

frequency $= \frac{1}{\text{period}} = \frac{\omega}{2\pi}$

Shift in phase $= \frac{\alpha}{\omega}$

Trig

$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

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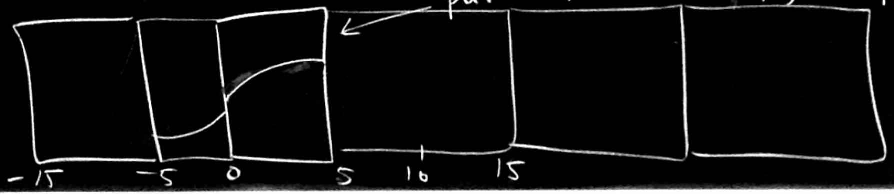
1(a) $f(x) = \sin(x+2x) = \sin(3x)$

put on rubber stamp, stamp left & right

1(b)



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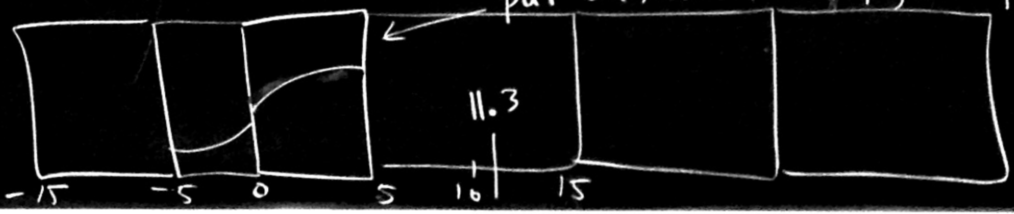

$f(10 + x) = f(x)$

$f(-10 + x) = f(x)$

$$f(11.3) = f(11.3-10) = f(1.3) = 100x\, e^{10x}\Big|_{x=1.3}$$

Theorem f, g have period p

- prod of periodic is periodic

  Quotient " " "

- h any function

  $\Rightarrow h(f(x))$ has period p

$$f(11.3) = f(11.3-10) = f(1.3) = 100x\, e^{10x}\Big|_{x=1.3}$$

1(c)

$\sin(\cos(2x))$	P	E	$\cancel{O}$
$\ln\lvert 2+\sin(x)\rvert$	P	$\cancel{E}$	$\cancel{O}$
$\sin(2x)\cos(x)$	P	$\cancel{E}$	O
$\dfrac{1+\sin(x)}{2+\cos(x)}$	P	$\cancel{E}$	$\cancel{O}$

Example

$\sin(x) + \sin(\pi x)$

Not periodic

Wave Equation, problem 1

DEF: Normal Modes = product Sols, Sols to $u_{tt} = c^2 u_{xx}$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

From problem 2

Eigenfunction

Eigenvalues $= \lambda^2, \lambda$

Wave Equation, problem 1

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From problem 2

Eigenfunction

Eigenvalues $= \lambda^2, \lambda$

Find sol $u(x,t)$.

Given

$u(x,0) = 0$ shape

$u_t(x,0) = -11\sin(5\pi x)$ speed

$$u(x,t) = (\text{constant}) \times (\text{normal Mode})$$

$$u = b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) = b_{10} \sin(5\pi x)\sin(5\pi c t)$$

$\uparrow$

$= 5\pi x$ when $L=2$

$$\frac{n\pi}{2} = 5\pi$$

$$n = 10$$

$u(x,t) = (\text{constant}) \times (\text{normal Mode})$

$$u = b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right) = b_{10} \sin(5\pi x) \sin(5\pi c t)$$

$\frac{n\pi x}{L} = 5\pi x$ when $L=2$

$\frac{n\pi}{2} = 5\pi$

$n = 10$

$u(x,0) = 0$ ✓

$u_t(x,0) = b_{10}(5\pi c)\sin(5\pi x)\cos(0) = -11\sin(5\pi x)$

$$\boxed{b_{10} = \frac{-11}{5\pi c}}$$

$u(x,t) = (\text{constant}) \times (\text{normal Mode})$

$u = b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right) = b_{10} \sin(5\pi x) \sin(5\pi c t)$

$\frac{n\pi x}{L} = 5\pi x$ when $L = 2$

$\frac{n\pi}{2} = 5\pi$

$n = 10$

answer

$$\boxed{u = \frac{-11}{5\pi c} \sin(5\pi x) \sin(5\pi c t)}$$

$u(x,0) = 0$ ✓

$u_t(x,0) = b_{10}(5\pi c) \sin(5\pi x) \cos(0) = -11 \sin(5\pi x)$

$$\boxed{b_{10} = \frac{-11}{5\pi c}}$$

Fourier Transform, problem 1

1ⓐ

$$F(\omega) = cv \int_{-\infty}^{\infty} f(x) e^{i\omega s x}\, dx \ , \quad cv = \frac{1}{2\pi}, \quad s = 1$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega s x}\, d\omega$$

$\int_{-\infty}^{\infty} |f(x)|\, dx$ finite

Example

$f(x) =$

$F(\omega) =$

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$$\int_{-\infty}^{\infty} |f(x)| dx \text{ finite}$$

Example

$$f(x) = \text{unit pulse} = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$F(\omega) = \ ?$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{Bmatrix} 1 & |x| < \frac{1}{2} \\ 0 & |x| \geq 1/2 \end{Bmatrix} e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi} \int_{-1/2}^{1/2} 1 \cdot e^{i\omega x}\, dx$$

$$= \left. \frac{e^{i\omega x}}{2\pi i\omega} \right|_{x=-1/2}^{x=1/2} = \frac{1}{2\pi i\omega}\left(e^{i\omega/2} - e^{-i\omega/2}\right)$$

Fourier Transform, problem 1

1 (a)

$$F(\omega) = cv \int_{-\infty}^{\infty} f(x) e^{i\omega s x} dx \,, \quad cv = \frac{1}{2\pi}, \quad s = 1$$

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Example

$$f(x) = \text{unit pulse} = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$F(\omega) = \frac{1}{\pi w} \sin\left(\frac{w}{2}\right)$$

Fourier Transform, problem 1

1(a)

$$F(\omega) = cv \int_{-\infty}^{\infty} f(x) e^{i\omega s x}\, dx \;, \quad cv = \frac{1}{2\pi}, \quad s = 1$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega s x}\, d\omega$$

1(b)

$$\begin{cases} u_t = k u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Transform The PDE

$$FT[u_t] = k\, FT[u_{xx}]$$

Apply Theorems, Rules for FT

t fixed $\longrightarrow \frac{\partial}{\partial t} FT[u(x,t)]$

$$FT[u_t(x,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(x,t)e^{i\omega x}\,dx$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx\right)$$

$$= \frac{\partial}{\partial t} FT[u(x,t)]$$

let $t = t_0$

$$FT[u_t(x,t_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(x,t_0)e^{i\omega x}\,dx$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t_0)e^{i\omega x}\,dx\right)$$

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Fourier Transform, problem 1

1(a)

$$F(\omega) = c_v \int_{-\infty}^{\infty} f(x) e^{i\omega s x}\, dx, \quad c_v = \frac{1}{2\pi}, \quad s = 1$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega s x}\, d\omega$$

1(b)

$$\begin{cases} u_t = k u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Transform The PDE

$$FT[u_t] = k\, FT[u_{xx}]$$

Apply Theorems, Rules for FT

$\underset{\text{fixed}}{t} \longrightarrow$ $$\frac{\partial}{\partial t} FT[u(x,t)] = k(-i\omega)(-i\omega) FT[u(x,t)]$$

$$FT\left[\frac{\partial v}{\partial x}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial v}{\partial x} e^{i\omega x} d_x$$

$$= uv - \int v\,du$$

$$= \frac{1}{2\pi}\left(e^{i\omega x} v \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v(i\omega) e^{i\omega x} d_x \right)$$

zero

$$= (-i\omega)\, FT[v]$$

$$\frac{d}{dt} U(t) = -kw^2 U(t)$$

where $U(t) = FT[u(x,t)]$

First order ODE

$$U = \frac{\text{constant}}{\text{integ factor}}$$

$$\boxed{U(t) = U_0 e^{-kw^2 t}}$$

$$\frac{d}{dt} U(t) = -kw^2 U(t)$$

where $U(t) = FT[u(x,t)]$

First order ODE

$$U = \frac{\text{constant}}{\text{integ factor}}$$

$$\boxed{U(t) = U_0 e^{-kw^2 t}}$$

Evaluate U_0

$$FT[u(x,t)] = U_0 e^{-kw^2 t}$$

$$FT[u(x,0)] = U_0 e^0$$

$$FT[f(x)] = U_0$$

$$F(\omega) = U_0$$

Fouriers Method Conclusion

$$FT[u(x,t)] = F(\omega)\, e^{-kw^2 t}$$

$$\uparrow \qquad\qquad \uparrow \quad \uparrow$$

$$U(t) \quad = \quad U_0\, e^{-kw^2 t}$$

$$\frac{d}{dt} U(t) = -kw^2 U(t)$$

where $U(t) = FT[u(x,t)]$

First order ODE

$$U = \frac{\text{constant}}{\text{integ factor}}$$

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Evaluate U_0

$$FT[u(x,t)] = U_0 e^{-kw^2 t}$$

$$FT[u(x,0)] = U_0 e^{0}$$

$$FT[f(x)] = U_0$$

$$F(\omega) = U_0$$

Fouriers Method Conclusion

$$FT[u(x,t)] = F(\omega)\, e^{-kw^2 t}$$

$$\uparrow \qquad \uparrow \quad \uparrow$$

$$U(t) = U_0\, e^{-kw^2 t}$$

Table entry

$$FT[u(x,t)] = F(\omega) G(\omega)$$

$$FT[u(x,t)] = FT[f * g] \quad \text{Conv thm}$$

$$u(x,t) = f * g = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) g(x-v)\, dv$$