Convolution Theorem

$$FT[f*g] = FT[f]\,FT[g]$$

or

$$FT[f*g](\omega) = F(\omega)G(\omega)$$

DEF: $$(f*g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\,g(x-v)\,dv$$

Transform Table, Gaussian

$f(x)$	$F(\omega)$
$e^{-\beta x^2}$	$\frac{1}{\sqrt{4\pi\beta}}\, e^{-\frac{\omega^2}{4\beta}}$
$\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$	$e^{-\alpha\omega^2}$

Calculus Review

$$\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}$$

polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad dx\,dy = r\,dr\,d\theta$$

Fubini interchange Thm

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

$$R = \left\{(x,y) : \begin{matrix} a\le x\le b \\ c\le y\le d\end{matrix}\right\} = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Calculus Theorem

$$\text{ans}^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy \qquad \text{Fubini}$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$2\pi\int_0^{\infty} r e^{-r^2}dr$$

$$\begin{cases} u = e^{-r^2} \\ du = -2re^{-r^2}dr \end{cases}$$

$$= 2\pi\int_1^0 \frac{du}{-2}$$

$$= 2\pi\int_0^1 \frac{du}{2}$$

$$= \pi$$

FRIDAY 28 Mar

Photo 2

Fourier Series and Fourier Transform

$f(x) = 0$ for $|x| > L$

$$a_n + i b_n = 2 c_n = \frac{\pi}{L} F\left(\frac{n\pi}{L}\right)$$

Fourier Coeff Complex Coeff Fourier Transform

Computer experiments

$$f(x) = \begin{cases} 20 \sin(9\pi x) & \text{on } -L < x < L,\ L \text{ large} \\ 0 & \text{else} \end{cases}$$

plot $|F(\omega)|$ and look for $\omega = 9\pi$. Then add more terms

Last Wed, 26 apr

Fact 2 $\quad f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$

$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$

satisfies

$f' + \frac{x}{2\alpha} f = 0, \quad f(0) = \sqrt{\frac{\pi}{\alpha}}$

Fact 1 $\quad$ Def: $g(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$

Satisfies $\quad g' + \frac{x}{2\alpha} g = 0, \quad g(0) = \sqrt{\frac{\pi}{\alpha}}$

Details

$$f'(x) = \frac{d}{dx}\int_{-\infty}^{\infty} F(w)\, e^{-iwx}\, dw$$

$$= \int_{-\infty}^{\infty} F(w)\, \frac{d}{dx}\left(e^{-iwx}\right) dx$$

$$= \int_{-\infty}^{\infty} F(w)(-iw)\, e^{-iwx}\, dx$$

Choose $F(w) = e^{-\alpha w^2}$

$$= \int_{-\infty}^{\infty} (-iw)\, e^{-\alpha w^2} e^{-iwx}\, dw$$

$$= uv - \int v\, du \qquad \left\{\begin{array}{l} u = i e^{-iwx} \\ dv = -w e^{-\alpha w^2} dw \\ du = -i^2 x\, e^{-iwx} dw \\ v = \frac{1}{2\alpha} e^{-\alpha w^2} \end{array}\right.$$

$$= i e^{-iwx} \frac{e^{-\alpha w^2}}{2\alpha}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{e^{-\alpha w^2}}{2\alpha} (x)\, e^{-iwx}\, dw$$

$$= 0$$

Details

$$f'(x) = \frac{d}{dx}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega$$
$$= \int_{-\infty}^{\infty} F(\omega)\frac{d}{dx}\left(e^{-i\omega x}\right)dx$$
$$= \int_{-\infty}^{\infty} F(\omega)(-i\omega)e^{-i\omega x}dx$$

Choose $F(\omega) = e^{-\alpha\omega^2}$

$$= \int_{-\infty}^{\infty}(-i\omega)e^{-\alpha\omega^2}e^{-i\omega x}d\omega$$

$$= uv - \int v\,du \qquad \begin{cases} u = ie^{-i\omega x} \\ dv = -\omega e^{-\alpha\omega^2}d\omega \\ du = -i^2 x e^{-i\omega x}d\omega \\ v = \frac{1}{2\alpha}e^{-\alpha\omega^2} \end{cases}$$

$$= ie^{-i\omega x}\frac{e^{-\alpha\omega^2}}{2\alpha}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{-\alpha\omega^2}}{2\alpha}(x)e^{-i\omega x}d\omega$$

$$= 0 - \frac{x}{2\alpha}\int_{-\infty}^{\infty}e^{-\alpha\omega^2}e^{-i\omega x}d\omega$$

$$= -\frac{x}{2\alpha}f(x)$$

$$f' + \frac{x}{2\alpha}f = 0$$

$$f(0) = \int_{-\infty}^{\infty}F(\omega)e^0 d\omega$$
$$= \int_{-\infty}^{\infty}e^{-\alpha\omega^2}d\omega$$
$$= \int_{-\infty}^{\infty}e^{-z^2}\frac{dz}{\sqrt{\alpha}}$$
$$= \frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

Choose

$z = \sqrt{\alpha}\,\omega$

$dz = \sqrt{\alpha}\,d\omega$

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$\sqrt{\frac{\pi}{\alpha}}e^{-\frac{x^2}{4\alpha}}$	$e^{-\alpha\omega^2}$

$\beta = \frac{1}{4\alpha}$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \, d\omega$$

$$FT[f](\omega) \equiv F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$

Theorem 1 $\quad \frac{\partial}{\partial t} FT[u(x,t)] = FT\left[\frac{\partial u}{\partial t}(x,t)\right]$

$\overline{\text{LHS}} = \frac{\partial}{\partial t} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$ where $f(x) = u(x,t)$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-i\omega x} dx$

$= \text{RHS}$

Theorem 2 $\quad FT\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega \; FT[u(x,t)]$

$\overline{\text{LHS}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,t) e^{-i\omega x} dx$

Theorem 1 $\frac{\partial}{\partial t} FT[u(x,t)] = FT\left[\frac{\partial u}{\partial t}(x,t)\right]$

$$LHS = \frac{\partial}{\partial t} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{where}$$

$$f(x) = u(x,t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-i\omega x} dx$$

$$= RHS$$

Theorem 2 $FT\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega \; FT[u(x,t)]$

$$LHS = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,t) e^{-i\omega x} dx$$

$$= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du \longleftarrow \left[\begin{array}{l} u = e^{-i\omega x} \\ dv = \frac{\partial u}{\partial x} dx \\ v = u(x,t) \\ du = (-i\omega) e^{-i\omega x} dx \end{array}\right.$$

$$= e^{-i\omega x} u(x,t)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega)\, u(x,t)\, e^{-i\omega x} dx$$

Theorem 1 $\quad \frac{\partial}{\partial t} FT[u(x,t)] = FT\left[\frac{\partial u}{\partial t}(x,t)\right]$

$$LHS = \frac{\partial}{\partial t} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{where} \quad f(x) = u(x,t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-i\omega x} dx$$

$$= RHS$$

assume: $u = 0$ at $x = \pm\infty$

Theorem 2 $\quad FT\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega \; FT[u(x,t)]$

$$LHS = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,t) e^{-i\omega x} dx$$

$$= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du \longleftarrow \left[\begin{array}{l} u = e^{-i\omega x} \\ dv = \frac{\partial u}{\partial x} dx \\ v = u(x,t) \\ du = (-i\omega) e^{-i\omega x} dx \end{array}\right.$$

$$= e^{-i\omega x} u(x,t)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega)\, u(x,t)\, e^{-i\omega x} dx$$

$$= 0 - (-i\omega)\; FT[u(x,t)]$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \, d\omega$$

$$FT[f](\omega) \equiv F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx$$

Theorem 1 $\frac{\partial}{\partial t} FT[u(x,t)] = FT\left[\frac{\partial u}{\partial t}(x,t)\right]$

$\underline{LHS} = \frac{\partial}{\partial t} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$ where $f(x) = u(x,t)$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{i\omega x} dx$

$= RHS$

assume:
$u = 0$ at $x = \pm\infty$

Theorem 2 $FT\left[\frac{\partial u}{\partial x}(x,t)\right] = -i\omega\, FT[u(x,t)]$

$\underline{LHS} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,t) e^{i\omega x} dx$

$= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du \longleftarrow \left[\begin{array}{l} u = e^{i\omega x} \\ dv = \frac{\partial u}{\partial x} dx \\ v = u(x,t) \\ du = (i\omega) e^{i\omega x} dx \end{array}\right.$

$= e^{i\omega x} u(x,t)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (i\omega) u(x,t) e^{i\omega x} dx$

$= 0 - (i\omega)\, FT[u(x,t)] = RHS$

$$\begin{cases} u_t = k\, u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) & \end{cases}$$

<u>Theorem</u> $u(x,t) =$ convolution of $f(x)$ and $g(x) = \frac{1}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}$

DEF: Heat Kernel $= \frac{1}{\sqrt{kt}} e^{-\frac{\quad}{4kt}}$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v)\, e^{-\frac{(x-v)^2}{4kt}} \frac{dv}{\sqrt{kt}}$$

$$FT[u_t] = k\,FT[u_{xx}]$$

$$\frac{\partial}{\partial t} FT[u] = k(i\omega)(i\omega)\, FT[u]$$

$\uparrow$ thm 1 $\qquad$ $\uparrow$ thm 2

$$\frac{dU}{dt} = -k\omega^2 U \quad , \quad U = FT[u(x,t)]$$

$$U = U_0 e^{-k\omega^2 t}$$

Determine U_0

put $t=0$

$$FT[u(x,0)] = U_0 e^0$$

$$FT[f(x)] = U_0$$

$$F(\omega) = U_0$$

$$U = F(\omega) e^{-k\omega^2 t}$$

$$U = F(\omega)\, e^{-\alpha\omega^2} \quad , \quad \alpha = kt$$

$$U = F(\omega)\, G(\omega) \qquad g(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

$$FT[u_t] = k\,FT[u_{xx}]$$

$$\frac{\partial}{\partial t} FT[u] = k(i\omega)(i\omega)\, FT[u]$$

thm 1   thm 2

$$\frac{dU}{dt} = -k\omega^2 U, \quad U = FT[u(x,t)]$$

$$U = U_0 e^{-k\omega^2 t}$$

Determine U_0

put $t=0$

$$FT[u(x,0)] = U_0 e^0$$

$$FT[f(x)] = U_0$$

$$F(\omega) = U_0$$

$$U = F(\omega) e^{-k\omega^2 t}$$

$$U = F(\omega) e^{-\alpha\omega^2}, \quad \alpha = kt$$

$$U = F(\omega) G(\omega) \qquad g(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

$$FT[u] = FT[f * g]$$

$$u = f * g$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\, g(x-v)\, dv =$$ ans on top board