H2.4-1(a) $u_t = \underset{\text{Equals } \frac{K_0}{c\rho}}{k}\, u_{xx}$, $u_x = 0$ at ends, $u(x,0) = f(x) = \begin{cases} 0 & 0 < x < L/2 \\ 1 & L/2 < x < L \end{cases}$

Answer: $$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{2p+1} \cos\left(\frac{(2p+1)\pi x}{L}\right) e^{-\frac{(2p+1)^2\pi^2 k t}{L^2}}$$

Eigenpairs: (λ_n, X_n), $\sqrt{\lambda_n} = \frac{n\pi}{L}$, $X_n = \cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, 3, \ldots$

New

H24-1(a)

$$u_t = k\, u_{xx}, \quad u_x = 0 \text{ at ends}, \quad u(x,0) = f(x) = \begin{cases} 0 & 0<x<L/2 \\ 1 & L/2<x<L \end{cases}$$

Equals $\frac{K_0}{c\rho}$

answer: $$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{2p+1}\cos\left(\frac{(2p+1)\pi x}{L}\right) e^{-\frac{(2p+1)^2\pi^2 kt}{L^2}}$$

Eigenpairs: (λ_n, X_n), $\sqrt{\lambda_n} = \frac{n\pi}{L}$, $X_n = \cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, 3, \ldots$

New

$$u(x,t) = \text{Superposition of prod sols} = \sum_{n=0}^{\infty} a_n X_n T_n = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

Have to compute $\{a_n\}_{n=0}^{\infty}$

Set $t=0$ to get

$$f(x) = \sum_{n=0}^{\infty} a_n X_n(x)$$

Use Method (1), 29 Jan

Mult by X_m for some $m = 0, 1, 2, \cdots$, Then integrate $0<x<L$

$$\int_0^L f X_m\, dx = \sum_{n=0}^{\infty} a_n \int_0^L X_n X_m\, dx \qquad m \text{ fixed}$$

$$= a_m \int_0^L X_m X_m\, dx \quad \leftarrow \text{no } \boxed{n}, \text{ only } m.$$

Orthogonality

when $n \neq m$, $\int_0^L X_n X_m \, dx = 0$

when $n = m$, $\int_0^L X_m^2 \, dx > 0$

H2.4-7 is Extra Credit

XC-H2.4-7

H2.4-6 has been edited

Dr. Will Nesse has

3150 videos ready in 1 week

$$a_m = \frac{\int_0^L f\, X_m\, dx}{\int_0^L X_m^2\, dx}$$

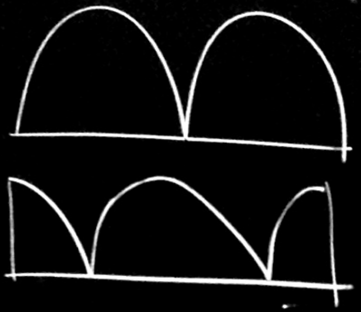
$$\boxed{a_m = \frac{2}{L}\int_0^L f\, X_m\, dx}$$

Orthogonality

when $n \neq m$, $\int_0^L X_n X_m\, dx = 0$

when $n = m$, $\int_0^L X_m^2\, dx > 0$

$\sin^2\theta$



$$\int_0^L \sin^2 = \frac{L}{2}$$

$$\int_0^L \cos^2 = \frac{L}{2}$$

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```
> int(1*cos(n*Pi*x/L),x=L/2..L) assuming n>0;
```

$$-\frac{L\left(\sin\left(\frac{1}{2}n\pi\right)-\sin(n\pi)\right)}{n\pi} \qquad (1)$$

```
>
```

Untitled (1)* - [Server 1] - Maple 15

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```
> int(1*cos(n*Pi*x/L),x=L/2..L) assuming n>0,
  n::integer;
```

$$-\frac{\sin\left(\frac{1}{2}\,n\,\pi\right)L}{\pi\,n} \qquad (1)$$

```
> 
```

Untitled (1)* - [Server 1] - Maple 15

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```
> int(1*cos(n*Pi*x/L),x=L/2..L) assuming n>0,
  n::integer;
```

$$-\frac{\sin\left(\frac{1}{2}n\pi\right)L}{\pi n} \qquad (1)$$

```
> int(1*cos(0*Pi*x/L),x=L/2..L);
```

$$\frac{1}{2}L \qquad (2)$$

```
>
```

Ready

$$a_m = \frac{2}{L}\int_0^L f\, X_m \, dx$$

$$= \frac{2}{L}\left(\underbrace{\int_0^{L/2}}_{\text{zero}} + \int_{L/2}^{L} \right)$$

$$= \frac{2}{L}\int_{L/2}^{L} 1 \cdot X_m \, dx$$

$$a_m = \begin{cases} \dfrac{L}{2} & m = 0 \\[2ex] \dfrac{(-1)^{p+1} L}{(2p+1)\pi} & n = 2p+1 \\[2ex] 0 & n = 2p \end{cases}$$

Facts

$\sin(n\pi) = 0$

$\cos(n\pi) = (-1)^n$

$\sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n$

$$\sum_{n=0}^{\infty} S_n = S_0 + \sum_{n=1}^{\infty} S_n = S_0 + \sum_{\substack{\text{even}\\ n=2p}} + \sum_{\substack{\text{odd}\\ n=2p+1}} = S_0 + \sum_{p=1}^{\infty} S_{2p} + \sum_{p=0}^{\infty} S_{2p+1}$$

Facts

$$\sin(n\pi) = 0$$

$$\cos(n\pi) = (-1)^n$$

$$\sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n$$

H2.4-1(b)

$$\underbrace{u_t = k\, u_{xx}}_{k \text{ Equals } \frac{K_0}{c\rho}}, \quad u_x = 0 \text{ at ends}, \quad u(x,0) = f(x) = \underbrace{6 + 4\cos\left(\frac{3\pi x}{L}\right)}_{\text{linear Combination of Eigenfunctions } X_n}$$

$$u(x,0) = f(x) = 6\,X_0 + 4\,X_3$$

$a_m = 0$ unless $m = 0$ or $m = 3$

$a_0 = 6$

$a_3 = 4$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n X_n e^{-\lambda_n k t} = a_0 + a_3 X_3 e^{-\lambda_3 k t} = 6 + 4\cos\left(\frac{3\pi x}{L}\right) e^{-\frac{9\pi^2 k t}{L^2}}$$

H 2.4-1(d)

$$f(x) = -3\cos\left(\frac{8\pi x}{L}\right) = -3\,X_8$$

answer: $$u(x,t) = -3\cos\left(\frac{8\pi x}{L}\right)e^{-\frac{64\pi^2 kt}{L^2}}$$

H2.4-1(c) $u_t = k u_{xx}$, $u_x = 0$ at ends, $u(x,0) = f(x) = -2\sin\left(\frac{\pi x}{L}\right)$

Equals $\frac{K_0}{c\rho}$

$$a_0 = \frac{1}{L}\int_0^L f \cdot 1\, dx = \frac{-4}{\pi}$$

$$a_1 = \frac{2}{L}\int_0^L f\, X_1\, dx = \frac{2}{L}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) dx = 0$$

$$a_m = \frac{2}{L}\int_0^L f\, X_m\, dx = \frac{2}{L}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{4\left((-1)^m + 1\right)}{\pi(m^2-1)}$$

answer:

$$u(x,t) = \frac{-4}{\pi} + \sum_{m=2}^{\infty} \frac{4\left((-1)^m + 1\right)}{\pi(m^2-1)} \cos\left(\frac{m\pi x}{L}\right) e^{-\frac{m^2\pi^2 kt}{L^2}}$$

H2.4-2 $u_t = k u_{xx}$, $u_x = 0$, $u = 0$ at $x=0$, $x=L$ resp

$u(x,0) = f(x)$

Prod Sol $u = X T$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0,\ X(L) = 0 \end{cases} \qquad \begin{cases} T' + \lambda k T = 0 \\ T(0) \neq 0 \end{cases}$$

Case $\lambda = 0$: $X'' = 0 \Rightarrow X = C_1 + C_2 x$

$X'(0) = 0 \Rightarrow \boxed{C_2 = 0}$

$X(L) = 0 \Rightarrow \boxed{C_1 = 0}$

$X = 0$

No eigenpair

Case $\lambda < 0$: Removed by Hint That T cannot have pos exponential

Case $\lambda > 0$

H2.4-2 $u_t = k u_{xx}$, $u_x = 0$, $u = 0$ at $x = 0$, $x = L$ resp

$u(x,0) = f(x)$

case $\lambda > 0$: $X = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$X'(0) = 0 \Rightarrow -\sqrt{\lambda}c_1\sin(0) + \sqrt{\lambda}c_2\cos(0) = 0$

$\boxed{c_2 = 0}$

$X(L) = 0 \Rightarrow c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L) = 0$

$\Rightarrow c_1\cos(\sqrt{\lambda}L) = 0$

$\Rightarrow \cos(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda}L =$ odd mult of $\frac{\pi}{2} = (2n+1)\frac{\pi}{2}$

Eigenpairs (λ_n, X_n), $\sqrt{\lambda_n} = \dfrac{(2n+1)\frac{\pi}{2}}{L}$, $X_n = \cos(\lambda_n x)$ $\quad n = 0, 1, 2, \ldots$

ans $\displaystyle u(x,t) = \sum_{m=0}^{\infty} a_m X_m T_m = \sum_{m=0}^{\infty} a_m \cos\left(\frac{(2m+1)\pi x}{L}\right) e^{-\frac{(2m+1)^2 \pi^2 kt}{L^2}}$

$$a_m = \frac{2}{L}\int_0^L f(x)\, X_m(x)\,dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2m+1)\pi x}{L}\right)dx$$

H2.4-3

$$X'' + \lambda X = 0$$

$$X(0) = X(2\pi)$$

$$X'(0) = X'(2\pi)$$

$\lambda = 0$: Eigenpair $\lambda = 0$, $X = 1$

$\lambda < 0$: NO Eigenpair.

$\lambda > 0$: Eigenpair

$\sqrt{\lambda} = n$, $X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$$2\pi\sqrt{\lambda} = 2\pi$$
$$= 4\pi$$

DETAILS BC $\Rightarrow$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sqrt{\lambda}\sin\theta & \sqrt{\lambda}(\cos\theta - 1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\theta = 2\pi\sqrt{\lambda}$

c_1, c_2 not both zero $\Leftrightarrow$ DET $= 0 \Leftrightarrow \cos(2\pi\sqrt{\lambda}) = 1$