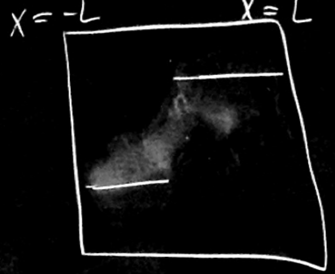
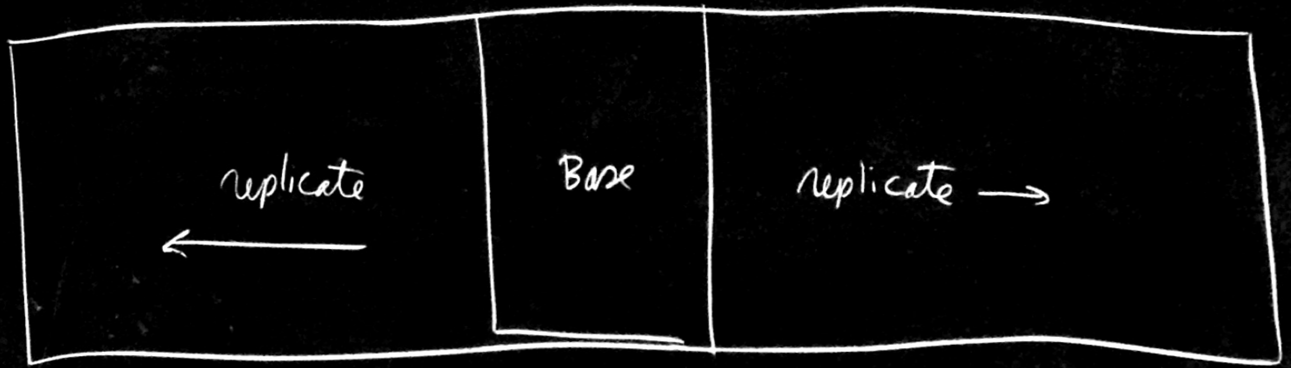
H3.3-1, 2, 3

Draw a periodic function

on $-\infty$ to ∞

Each Series is 2L-periodic





Fourier Series $\quad a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$

$f(x)$ on $[-L, L]$

Fourier Sine Series $\quad \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$f(x)$ on $[0, L]$

Fourier Cosine Series $\quad a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

$f(x)$ on $[0, L]$

Fourier Series $\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f \qquad a_n = \frac{2}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{2}{2L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

F. Sine Series : $\quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

F. Cosine Series : $\quad a_0 = \frac{1}{L}\int_{0}^{L} f, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

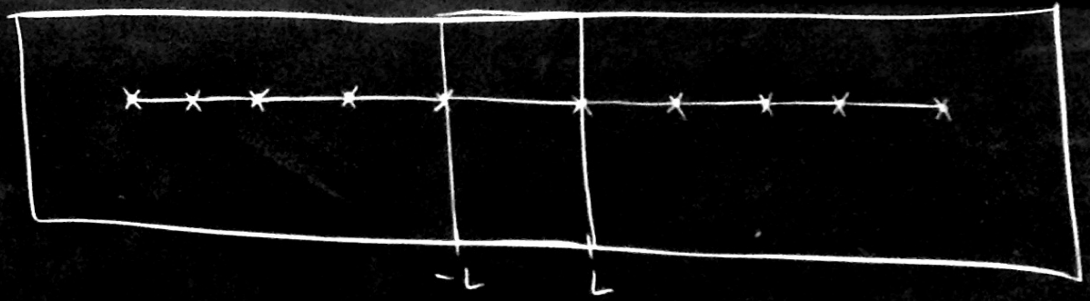
H3.3-1a

Fourier Series

$f(x)=1$ on $0<x<L$

or

$f(x)=1$ on $-L<x<L$



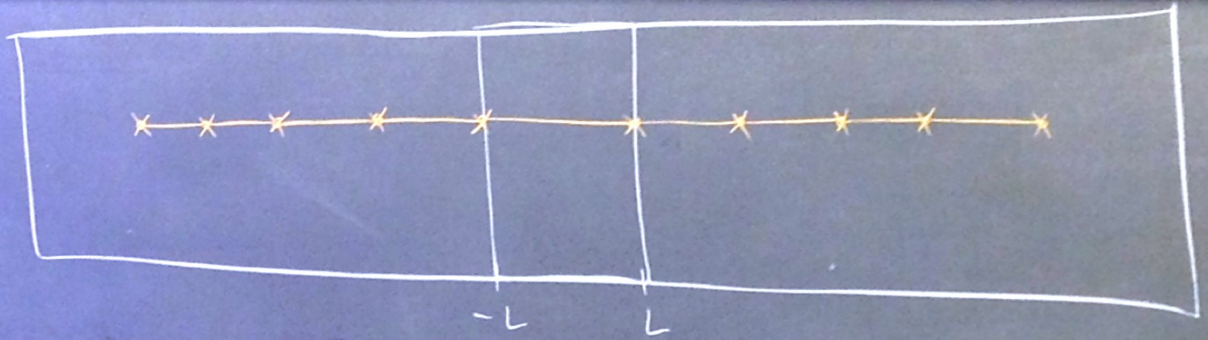
H 3.3-1a

Fourier Series

$f(x) = 1$ on $0 < x < L$

or

$f(x) = 1$ on $-L < x < L$



H3.3-1a

$f(x) = 1$ on $0 < x < L$

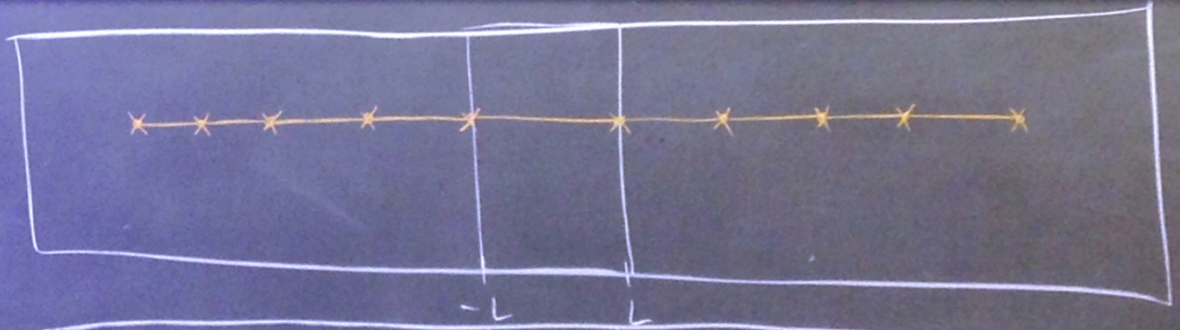
or

$f(x) = 1$ on $-L < x < L$

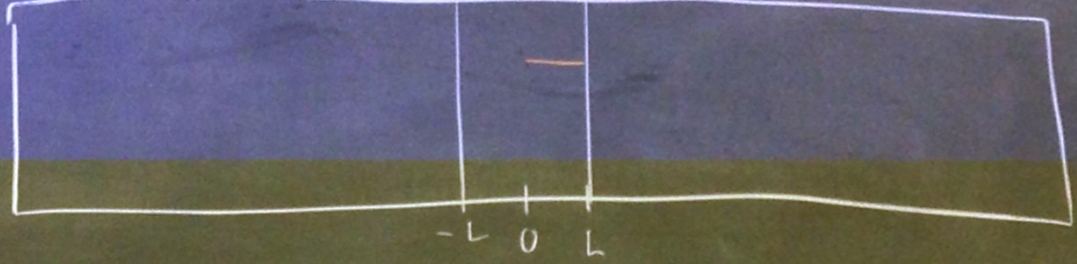
Fourier Series

&

Fourier Cosine Series



Fourier Sine Series



H3.3-1a

$f(x) = 1$ on $0 < x < L$

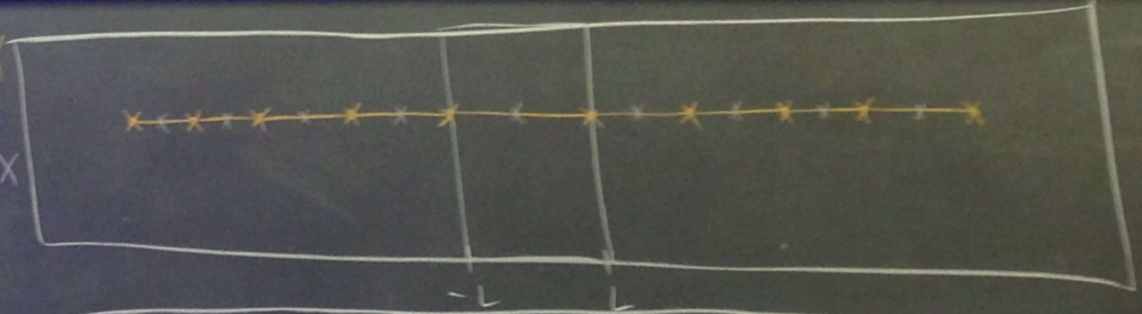
or

$f(x) = 1$ on $-L < x < L$

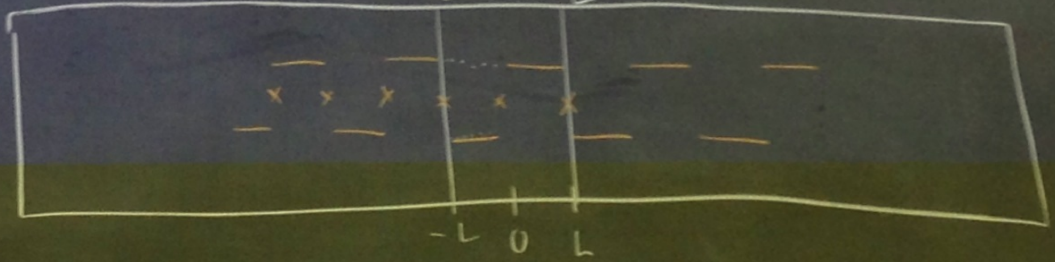
Fourier Series X

&

Fourier Cosine Series X



Fourier Sine Series



Fourier Sine Series =

Regular Fourier Series for

$$f_1(x) = \begin{cases} f(x) & 0 < x < \\ -f(-x) & -L < x < 0 \end{cases}$$

= odd extension of f

H 3.3 - 1b

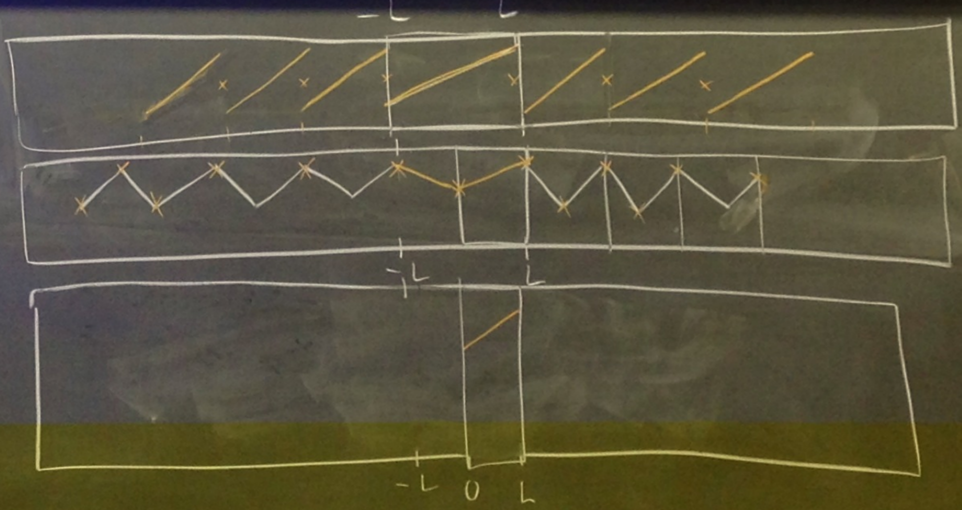
$f(x) = 1+x$ on $0<x<L$

or

$f(x) = 1+x$ on $-L<x<L$

Fourier Series & Fourier Cosine Series

Fourier Sine Series



H 3.3 - 1b

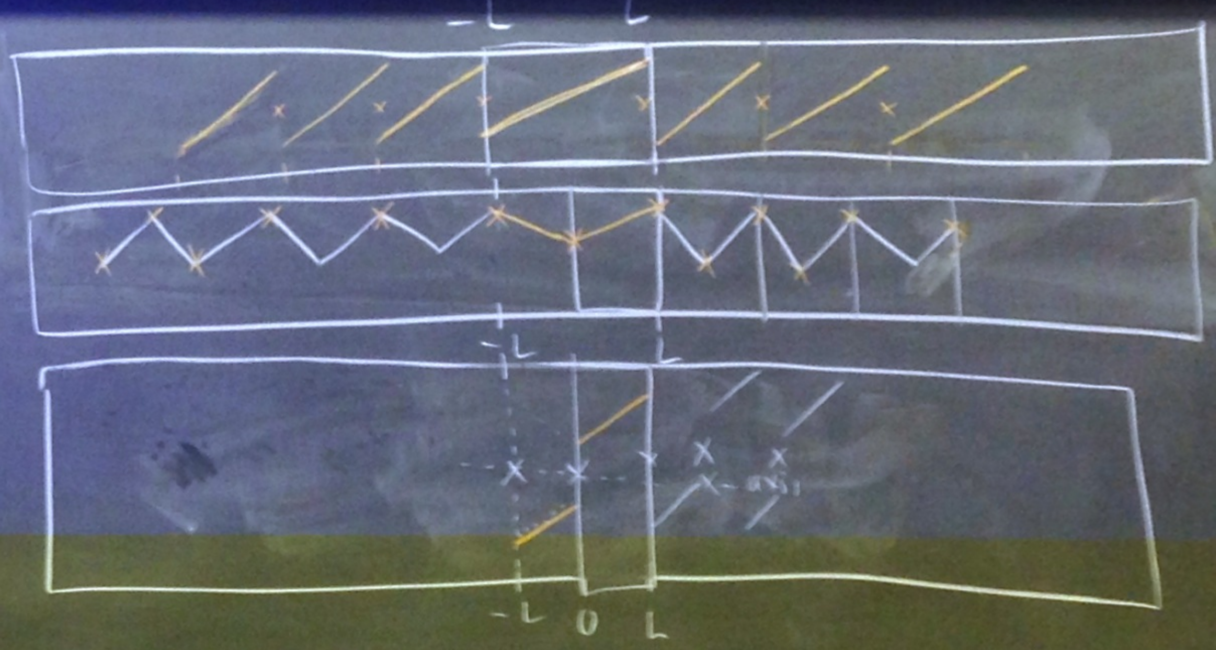
$f(x) = 1+x$ on $0 < x < L$

or

$f(x) = 1+x$ on $-L < x < L$

Fourier Series & Fourier Cosine Series

Fourier Sine Series



$0<x<L$

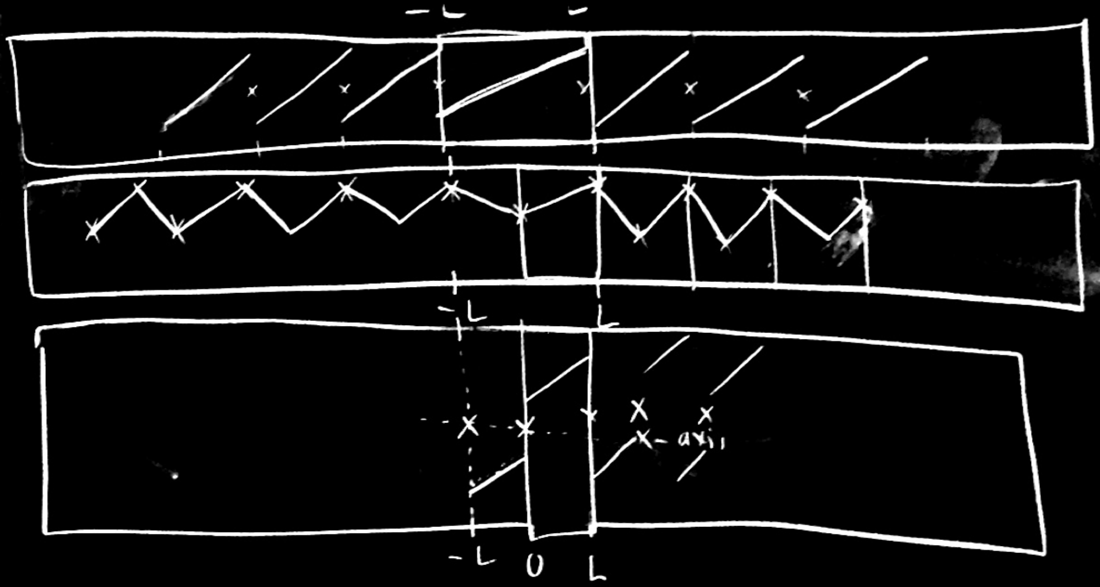
$-L<x<L$

Fourier Series

&

Fourier Cosine Series

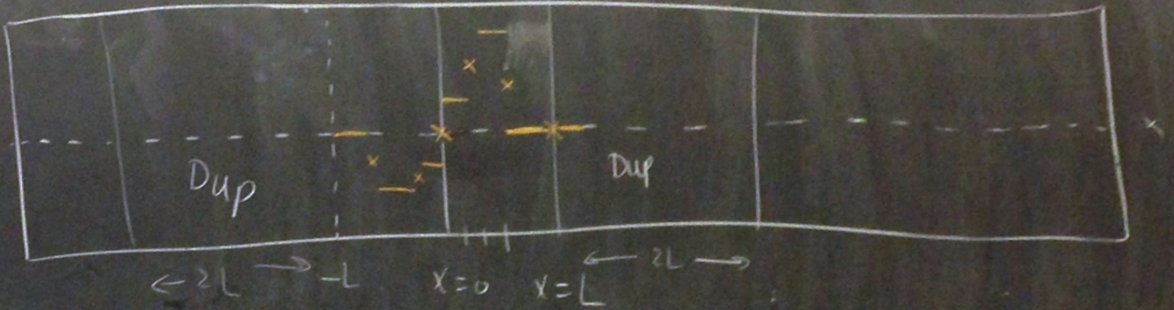
Fourier Sine Series



H3.3-2b

$$f(x)=\begin{cases} 1 & 0<x<\frac{L}{6} \\ 3 & \frac{L}{6}<x<\frac{L}{2} \\ 0 & \frac{L}{2}<x<L \end{cases}$$

Sine Series



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```
> # Find Fourier cosine coefficients for
  # f(x)=1 on L/2<x<L, f(x)=0 elsewhere
> L:=1;
```

$$L := 1 \qquad (1)$$

```
> f:=x->piecewise(x<L/2,0, x<L,1,0);
```

$$f := x \rightarrow \mathit{piecewise}\left(x < \frac{1}{2}L, 0, x < L, 1, 0\right) \qquad (2)$$

```
> convert(f(x),piecewise,x);
```

$$\begin{cases} 0 & x < \frac{1}{2} \\ 1 & x < 1 \\ 0 & 1 \le x \end{cases} \qquad (3)$$

```
> A:=n->(2/L)*int(f(x)*cos(n*Pi*x/L),x=0..L);
```

C:\texGrant\3150\3150Spring2014\references\blackboardphotos\26feb2014\H3.3-6b-fourier-cosine-coefficients-and-plot.mw - [Server 1] - Maple 15

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```
> L:=1;
```

$$L := 1 \qquad (1)$$

```
> f:=x->piecewise(x<L/2,0, x<L,1,0);
```

$$f := x \to piecewise\left(x < \frac{1}{2}L, 0, x < L, 1, 0\right) \qquad (2)$$

```
> convert(f(x),piecewise,x);
```

$$\begin{cases} 0 & x < \frac{1}{2} \\ 1 & x < 1 \\ 0 & 1 \leq x \end{cases} \qquad (3)$$

```
> A:=n->(2/L)*int(f(x)*cos(n*Pi*x/L),x=0..L);
```

$$2\left(\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

H3.3-2d-fourier-coefficients.pdf - Adobe Reader

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```
# f(x)=1 on x<L/6, f(x)=3 on L/6<x<L/2, f(x)=0 elsewhere
> L:=1;
```

$$L := 1 \qquad (1)$$

```
> f:=x->piecewise(x<0,0, x<L/6,1,x<L/2, 3, 0);
```

$$f := x \rightarrow \mathit{piecewise}\left(x < 0, 0, x < \frac{1}{6} L, 1, x < \frac{1}{2} L, 3, 0\right) \qquad (2)$$

```
> convert(f(x),piecewise,x);
```

$$\begin{cases} 0 & x < 0 \\ 1 & x < \frac{1}{6} \\ 3 & x < \frac{1}{2} \\ 0 & \frac{1}{2} \le x \end{cases} \qquad (3)$$

```
> # Truncated sine series S(x,n)
> B:=n->eval((2/L)*int(f(x)*sin(n*Pi*x/L),x=0..L));
```

$$\left(\frac{2}{L} \left(\int_0^L f(x) \sin\left(\frac{n \pi x}{L}\right) dx \right) \right)$$

H3.3-2d-fourier-coefficients.pdf - Adobe Reader

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$$\begin{cases} 3 & x < \frac{1}{2} \\ 0 & \frac{1}{2} \le x \end{cases} \tag{3}$$

```
> # Truncated sine series S(x,n)
> B:=n->eval((2/L)*int(f(x)*sin(n*Pi*x/L),x=0..L));
```

$$B := n \rightarrow eval\left(\frac{2\left(\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}{L}\right) \tag{4}$$

```
> B(1);
```

$$-\frac{-2+\sqrt{3}}{\pi}+\frac{3\sqrt{3}}{\pi} \tag{5}$$

```
> B(n) assuming n >0;
```

$$-\frac{2\left(-1+\cos\left(\frac{1}{6}n\pi\right)\right)}{n\pi}+\frac{6\left(\cos\left(\frac{1}{6}n\pi\right)-\cos\left(\frac{1}{2}n\pi\right)\right)}{n\pi} \tag{6}$$

H3.3-3b-fourier-coefficients-and-plot.pdf - Adobe Reader

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```
> B:=n->eval((2/L)*int(f(x)*sin(n*Pi*x/L),x=0..L));
```

$$B := n \rightarrow eval\left(\frac{2\left(\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}{L}\right) \tag{4}$$

```
> B(1);B(n) assuming n >0;
```

$$\frac{2}{\pi}$$

$$-\frac{2\left(-1+\cos\left(\frac{1}{2}n\pi\right)\right)}{n\pi} \tag{5}$$

```
> # Truncated sine series S(x,n)
> S:=(x,n)->sum(B(k)*sin(k*Pi*x/L),k=1..n);plot(S(x,50),x=-5*L..5*
  L);
```

$$S := (x, n) \rightarrow \sum_{k=1}^{n} B(k)\sin\left(\frac{k\pi x}{L}\right)$$

H3.3-3b-fourier-coefficients-and-plot.pdf - Adobe Reader

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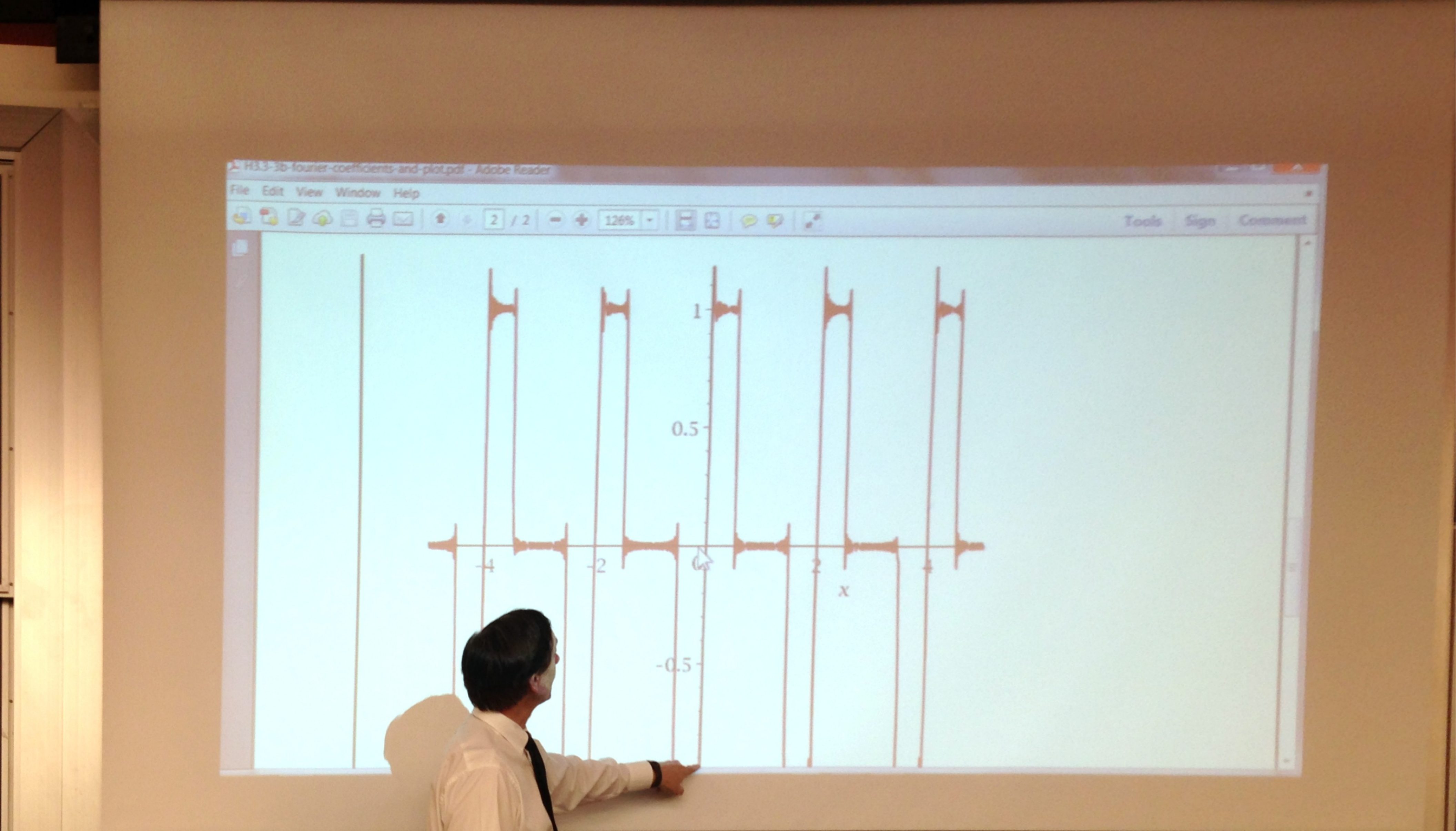
```
> B(1);B(n) assuming n >0;
```

$$\frac{2}{\pi}$$

$$-\frac{2\left(-1+\cos\left(\frac{1}{2}n\pi\right)\right)}{n\pi} \tag{5}$$

```
> # Truncated sine series S(x,n)
> S:=(x,n)->sum(B(k)*sin(k*Pi*x/L),k=1..n);plot(S(x,50),x=-5*L..5*
L);
```

$$S := (x, n) \to \sum_{k=1}^{n} B(k) \sin\left(\frac{k\pi x}{L}\right)$$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-2
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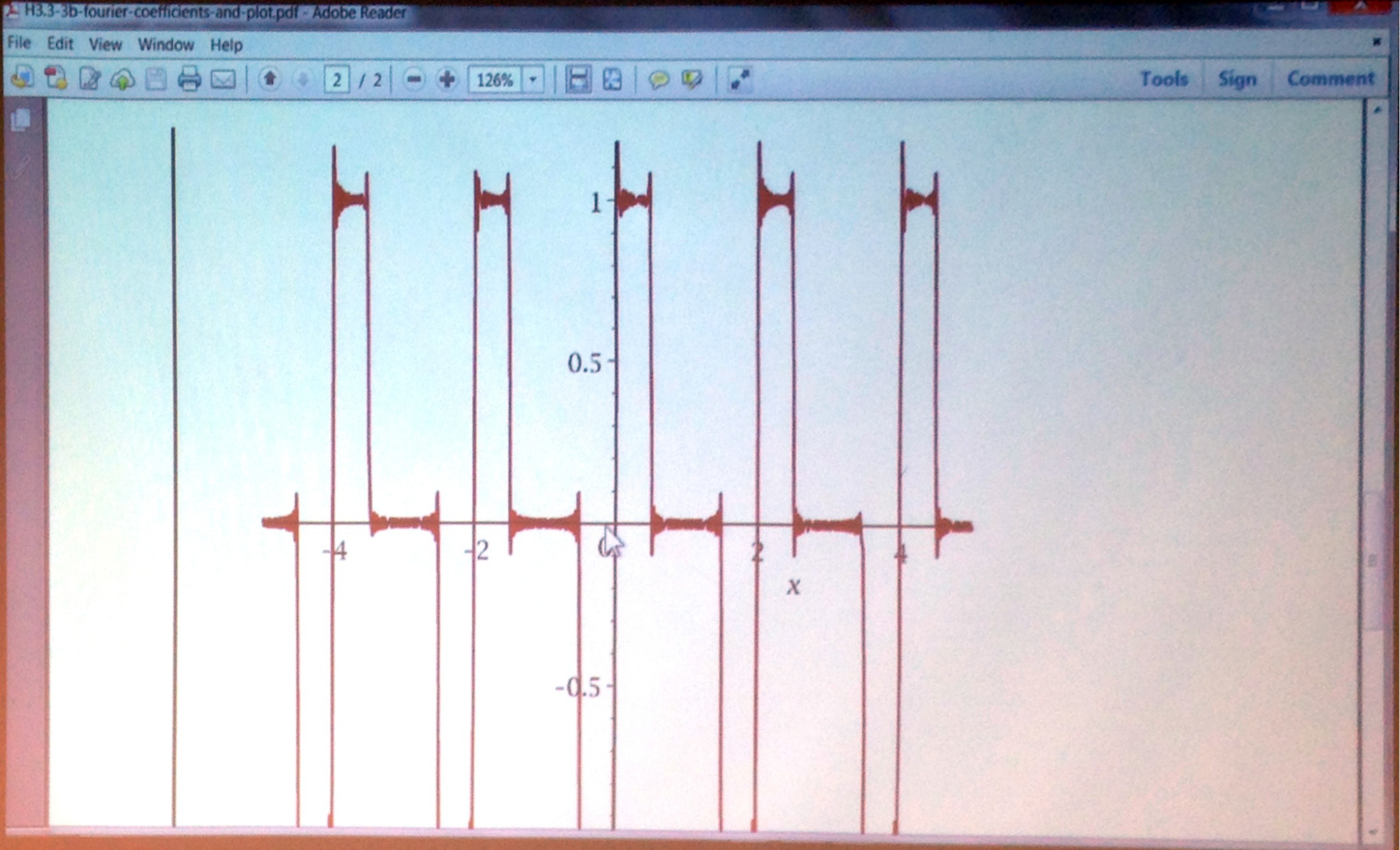
x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```
> B:=n->eval((2/L)*int(f(x)*sin(n*Pi*x/L),x=0..L));
```

$$B := n \rightarrow eval\left(\frac{2\left(\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)}{L}\right) \qquad (4)$$

```
> B(1);B(n) assuming n >0;
```

$$\frac{2}{\pi}$$

$$-\frac{2\left(-1+\cos\left(\frac{1}{2}n\pi\right)\right)}{n\pi} \qquad (5)$$

```
> # Truncated sine series S(x,n)
> S:=(x,n)->sum(B(k)*sin(k*Pi*x/L),k=1..n);plot(S(x,50),x=-5*L..5*
  L);
```

$$S := (x, n) \rightarrow \sum_{k=1}^{n} B(k)\sin\left(\frac{k\pi x}{L}\right)$$

H3.3-6b-fourier-cosine-coefficients-and-plot.pdf - Adobe Reader

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```
# f(x)=1 on L/2<x<L, f(x)=0 elsewhere
> L:=1;
```

$$L := 1 \tag{1}$$

```
> f:=x->piecewise(x<L/2,0, x<L,1,0);
```

$$f := x \to piecewise\left(x < \frac{1}{2}L, 0, x < L, 1, 0\right) \tag{2}$$

```
> convert(f(x),piecewise,x);
```

$$\begin{cases} 0 & x < \frac{1}{2} \\ 1 & x < 1 \\ 0 & 1 \le x \end{cases} \tag{3}$$

```
> A:=n->(2/L)*int(f(x)*cos(n*Pi*x/L),x=0..L);
```

$$A := n \to \frac{2\left(\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)}{L} \tag{4}$$

```
> A0:=(1/L)*int(f(x),x=0..L);
```

$$A0 := \frac{1}{2} \tag{5}$$

H3.3-6b-fourier-cosine-coefficients-and-plot.pdf - Adobe Reader

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$$0 \qquad 1 \le x$$

```
> A:=n->(2/L)*int(f(x)*cos(n*Pi*x/L),x=0..L);
```

$$A := n \rightarrow \frac{2\left(\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right)}{L} \qquad (4)$$

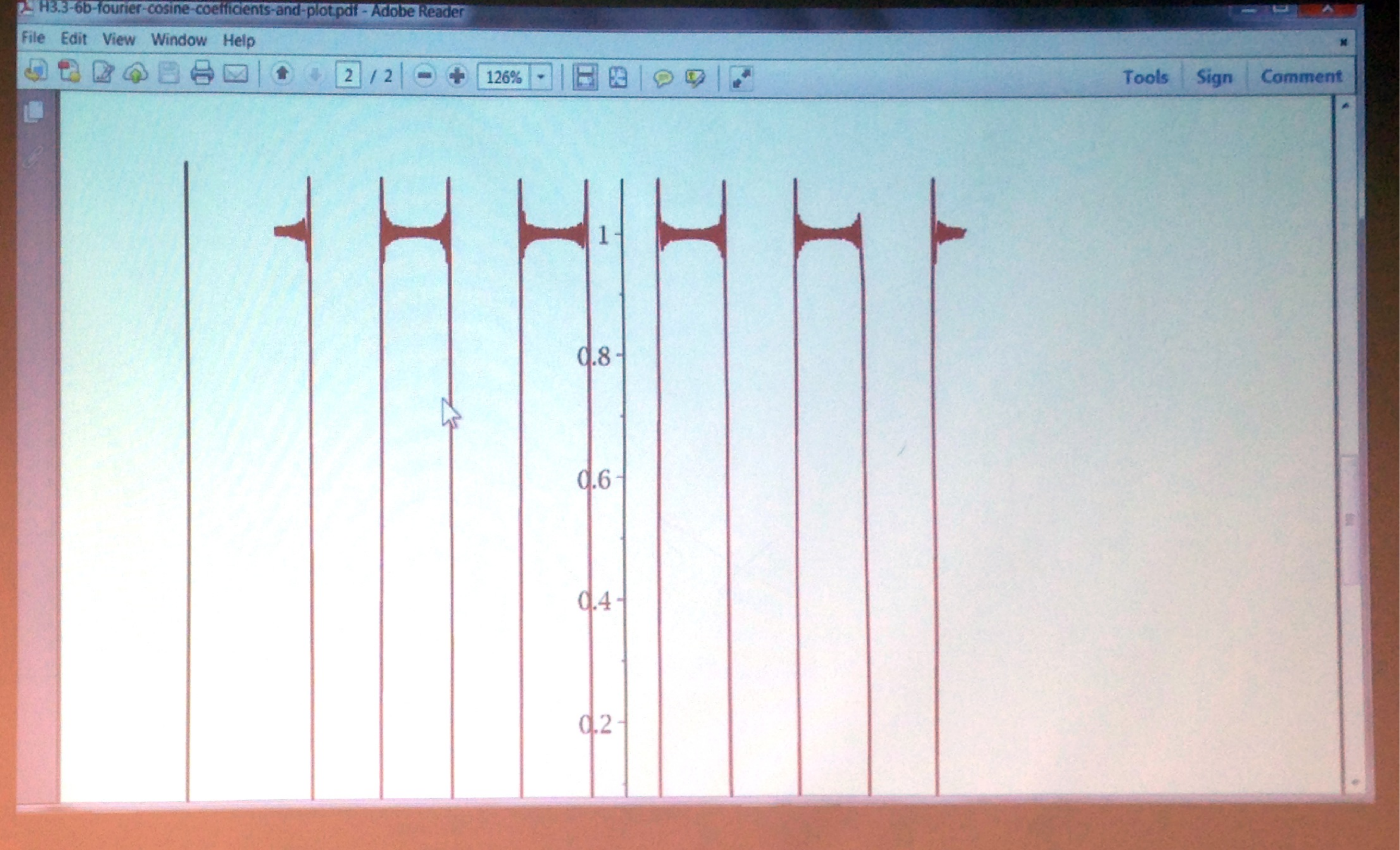
```
> A0:=(1/L)*int(f(x),x=0..L);
```

$$A0 := \frac{1}{2} \qquad (5)$$

```
> # Truncated cosine series T(x,n)
> T:=(x,n)->A0+sum(A(k)*cos(k*Pi*x/L),k=1..n);
```

$$T := (x, n) \rightarrow A0 + \sum_{k=1}^{n} A(k)\cos\left(\frac{k\pi x}{L}\right) \qquad (6)$$

```
> plot(T(x,50),x=-5*L..5*L);
```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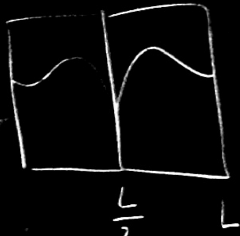
H 3.3-13 Even functions and Zero Fourier Coefficients

assume: $f(u+\frac{L}{2})$ even on $-\frac{L}{2}$ to $\frac{L}{2}$

Show: $b_{2k}=0$ in Fourier Sine Series



$\boxed{n=2k}$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{-L/2}^{L/2} f\left(u+\frac{L}{2}\right)\sin\left(\frac{n\pi u}{L}+\frac{n\pi L/2}{L}\right)dx$$

Let $x = u+\frac{L}{2}$

$dx = du$

x	u
0	$-L/2$
L	$L/2$

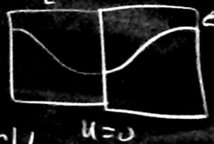
$$\sin(a+b) = \sin(a)\underbrace{\cos(b)}_{(-1)^k} + \underbrace{\sin(b)}_{\text{zero}}\cos(a)$$

$$a = \frac{n\pi u}{L}, \quad b = \frac{n\pi L/2}{L} = \frac{n\pi}{2} = \frac{2k\pi}{2} = k\pi$$

$$\sin\left(\frac{n\pi u}{L} + \frac{n\pi L/2}{L}\right) = (-1)^k \sin\left(\frac{n\pi u}{L}\right)$$

Trick:

$$b_n = \frac{2}{L}(-1)^k \int_{-L/2}^{L/2} f\left(u+\frac{L}{2}\right)\sin\left(\frac{n\pi u}{L}\right)du$$



$\leftarrow f(u+\frac{L}{2})$

Even function

$$b_n = \frac{2}{L}(-1)^k \int_{-L/2}^{L/2} (\text{even})(\text{odd})\,du$$

$$\sin(a+b) = \sin(a)\overset{(-1)^k}{\cos(b)} + \overset{\text{zero}}{\sin(b)}\cos(a)$$

$$a = \frac{n\pi u}{L}, \quad b = \frac{n\pi L/2}{L} = \frac{n\pi}{2} = \frac{2k\pi}{2} = k\pi$$

$$\sin\left(\frac{n\pi u}{L} + \frac{n\pi L/2}{L}\right) = (-1)^k \sin\left(\frac{n\pi u}{L}\right)$$

Trick:

$$b_n = \frac{2}{L}(-1)^k \int_{-L/2}^{L/2} f\left(u+\frac{L}{2}\right)\sin\left(\frac{n\pi u}{L}\right)du$$

$$b_n = \frac{2}{L}(-1)^k \int_{-L/2}^{L/2} (\text{even})(\text{odd})\,du = \frac{2}{L}(-1)^k \int_{-L/2}^{L/2} (\text{odd})\,du$$

$$= \boxed{\text{zero}}$$

H3.3-16 Derive Fourier Coefficient formulas for Fourier Series on $a \le x \le b$.

Hint: $y = \frac{a+b}{2} + \frac{b-a}{2}\frac{x}{L}$

Solve for x in terms of y

$$2y = a+b + (b-a)\frac{x}{L}$$

$$\frac{2y}{b-a} = \frac{a+b}{b-a} + \frac{x}{L}$$

$$x = L\left(\frac{2y}{b-a} - \frac{a+b}{b-a}\right)$$

H3.3-16 Derive Fourier Coefficient formulas for Fourier Series on $a \le x \le b$.

Hint: $y = \dfrac{a+b}{2} + \dfrac{b-a}{2}\dfrac{x}{L}$

Solve for x in terms of y

$$2y = a+b + (b-a)\frac{x}{L}$$

$$\frac{2y}{b-a} = \frac{a+b}{b-a} + \frac{x}{L}$$

$$x = L\left(\frac{2y}{b-a} - \frac{a+b}{b-a}\right)$$

$$\frac{n\pi x}{L} = \frac{2n\pi y}{b-a} - \frac{(a+b)n\pi}{b-a}$$

Fourier Series $= a_0 + \sum_1^\infty a_n \cos\left(\frac{2n\pi y}{b-a} - \left(\frac{a+b}{b-a}\right)n\pi\right) + b_n \sin(\text{same}) = \frac{f(y+)+f(y-)}{2} \overset{\text{Mostly}}{=} f(y)$

Write a_0 in terms of a, b

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_1(x)\,dx = \frac{1}{2L}\int_a^b f(y)\,\frac{2L}{b-a}\,dy$$

$$a_0 = \frac{1}{b-a}\int_a^b f(y)\,dy$$

x	y
L	b
$-L$	a

$f_1 = f$ image under $y \mapsto x$.

$f_1(x) = f(y)$

$$\begin{cases} y = \frac{a+b}{2} + \frac{b-a}{2}\frac{x}{L} \\ dy = \frac{b-a}{2}\frac{dx}{L} \end{cases}$$