Theorem

H3.4

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

$\Rightarrow \sqrt{\lambda} L = n\pi \quad n = 1, 2, 3, \ldots$

$\lambda > 0, \quad X = \sin(\sqrt{\lambda} x)$

Theorem 2.

H3.4

$$X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0$$

$\Rightarrow \lambda = 0$ and $X = 1$; or $\lambda > 0$,

$\sqrt{\lambda} L = n\pi$ and $X = \cos(\sqrt{\lambda} L)$

Def: $(\lambda, X) =$ Eigenpair

$\lambda =$ Eigenvalue

$X =$ Eigenfunction

H2.3 part 5

Product Solutions + Superposition

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$u = X(x)\,T(t)$

$X = \sin\left(\frac{n\pi x}{L}\right)$

$T = e^{-\frac{n^2\pi^2}{L^2}kt}$

Last Wed.

Superposition $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$

$$u_t = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

$$= \sum^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{-n^2\pi^2}{L^2}k\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

$$k u_{xx} = -k \sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

$x = 0$: $$u(0,t) = \sum_{n=1} b_n \sin\left(\frac{n\pi(0)}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

$$= \sum_{n=1}^{\infty} b_n (0) e^{-\frac{n^2\pi^2}{L^2}kt}$$

$$= 0$$

$x = L$: $$u(L,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi) e^{-\frac{n^2\pi^2}{L^2}kt} = 0$$

Last BC

$u(x,0) = f(x)$

$f(x) = u(x,0) = \sum_{n=1} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-0}$

$\boxed{\text{EXAMPLE}}$: $f(x) = \sin\left(\frac{\pi x}{L}\right) - 10 \sin\left(\frac{5\pi x}{L}\right) =$ Series

$b_1 = 1$

$b_5 = -10$

$b_n = 0$ for $n \neq 1,\ n \neq 5$

$$\boxed{u(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi}{L}\right)^2 kt} - 10 \sin\left(\frac{5\pi x}{L}\right) e^{-\left(\frac{5\pi}{L}\right)^2 kt}}$$

$$u = b_1 \sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi}{L}\right)^2 kt}$$

$$+ b_2 \sin\left(\frac{2\pi x}{L}\right) e^{-\left(\frac{2\pi}{L}\right)^2 kt}$$

$$+ \cdots$$

How to find NONZERO b_n s?

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$\vec{f}$ = package made from $y = f(x)$

$\vec{S_n}$ = " " " $y = \sin\left(\frac{n\pi x}{L}\right)$

$$\vec{f} = \sum_{n=1}^{\infty} b_n \vec{S_n}$$

DEF: $A(x)$, $B(x)$
functions on
$0 < x < L$

Then $\vec{A} \cdot \vec{B} = \int_0^L A(x)B(x)\,dx$

Appendix Haberman on orthogonality

$\vec{S}_1, \vec{S}_2, \vec{S}_3 \perp$ given

$\vec{X} = b_1 \vec{S}_1 + b_2 \vec{S}_2 + b_3 \vec{S}_3 \ldots$

$$b_1 = \frac{\vec{X} \cdot \vec{S}_1}{\vec{S}_1 \cdot \vec{S}_1}$$

Solve For b_1

only 1 non zero Term on RHS

Def: $A(x)$, $B(x)$ functions on $0 < x < L$

The $\vec{A} \cdot \vec{B} = \int_0^L A(x)B(x)\,dx$

Thus $b_2 = \dfrac{\vec{X} \cdot \vec{S}_2}{\vec{S}_2 \cdot \vec{S}_2}$

$$\vec{X} \cdot \vec{S}_1 = b_1 \underbrace{\vec{S}_1 \cdot \vec{S}_1}_{\text{non zero}} + b_2 \underbrace{\vec{S}_2 \cdot \vec{S}_1}_{\text{zero}} + b_3 \vec{S}_3 \cdot \vec{S}_1$$

$$b_3 = \frac{\vec{X} \cdot \vec{S}_3}{\vec{S}_3 \cdot \vec{S}_3}$$

Appendix
Haberman
Orthogonality

$\vec{S}_1, \vec{S}_2, \vec{S}_3 \perp$ Given

$\vec{x} = b_1 \vec{S}_1 + b_2 \vec{S}_2 + b_3 \vec{S}_3$

$$\boxed{b_n = \frac{\vec{f} \cdot \vec{S}_n}{\vec{S}_n \cdot \vec{S}_n}}$$

$f(x) = b_1 \sin(\pi x) + b_2 \sin(2\pi x) + b_3 \sin(3\pi x)$

Find b_1, b_2, b_3 by Calculus

Mult by $\sin(\pi x)$, integrate over $0 < x < L = 1$

$$\int_0^1 f(x)\sin(\pi x)\,dx = b_1\int_0^1 \sin^2(\pi x)\,dx + b_2\int_0^1 \sin(2\pi x)\sin(\pi x)\,dx + b_3\int_0^1 \sin(3\pi x)\sin(\pi x)\,dx$$

" $=$ NONzero $+$ zero $+$ zero

EXAMPLE

$u_t = u_{xx} \quad 0<x<1$

$u=0$ at $x=0,1$

$u(x,0) = \sin(5\pi x)$

$u = \sin(5\pi x)\, e^{-(5\pi)^2 t}$

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$\vec{f}$ = package made from $y = f(x)$

$\vec{S_n}$ = " " " $y = \sin\left(\frac{n\pi x}{L}\right)$

$\vec{f} = \sum_{n=1}^{\infty} b_n \vec{S_n}$

??