Fourier Sine Series

$$F_{sine}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Cosine Series

$$F_{cosine}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

$$100 = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad 0 < x < L = 1$$

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \\ u = 0 \text{ at } x = 0 \text{ and } x = 1 \\ u(x,0) = 100 \end{cases}$$

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Maple Input Courier New 12

```
> B:=k->(400/Pi)/(2*k+1);
```

$$B := k \rightarrow \frac{400}{\pi\,(2\,k+1)} \tag{1}$$

```
> g:=(x,n)->sum(B(k)*sin((2*k+1)*x),'k'=0..n);
```

$$g := (x, n) \rightarrow \sum_{'k'=0}^{n} B(k)\,\sin(\,(2\,k+1)\,x) \tag{2}$$

```
> g(x,1);
```

$$\frac{400\,\sin(x)}{\pi} + \frac{400}{3}\,\frac{\sin(3\,x)}{\pi} \tag{3}$$

```
> plot(g(x,1),x=-Pi..Pi);
```

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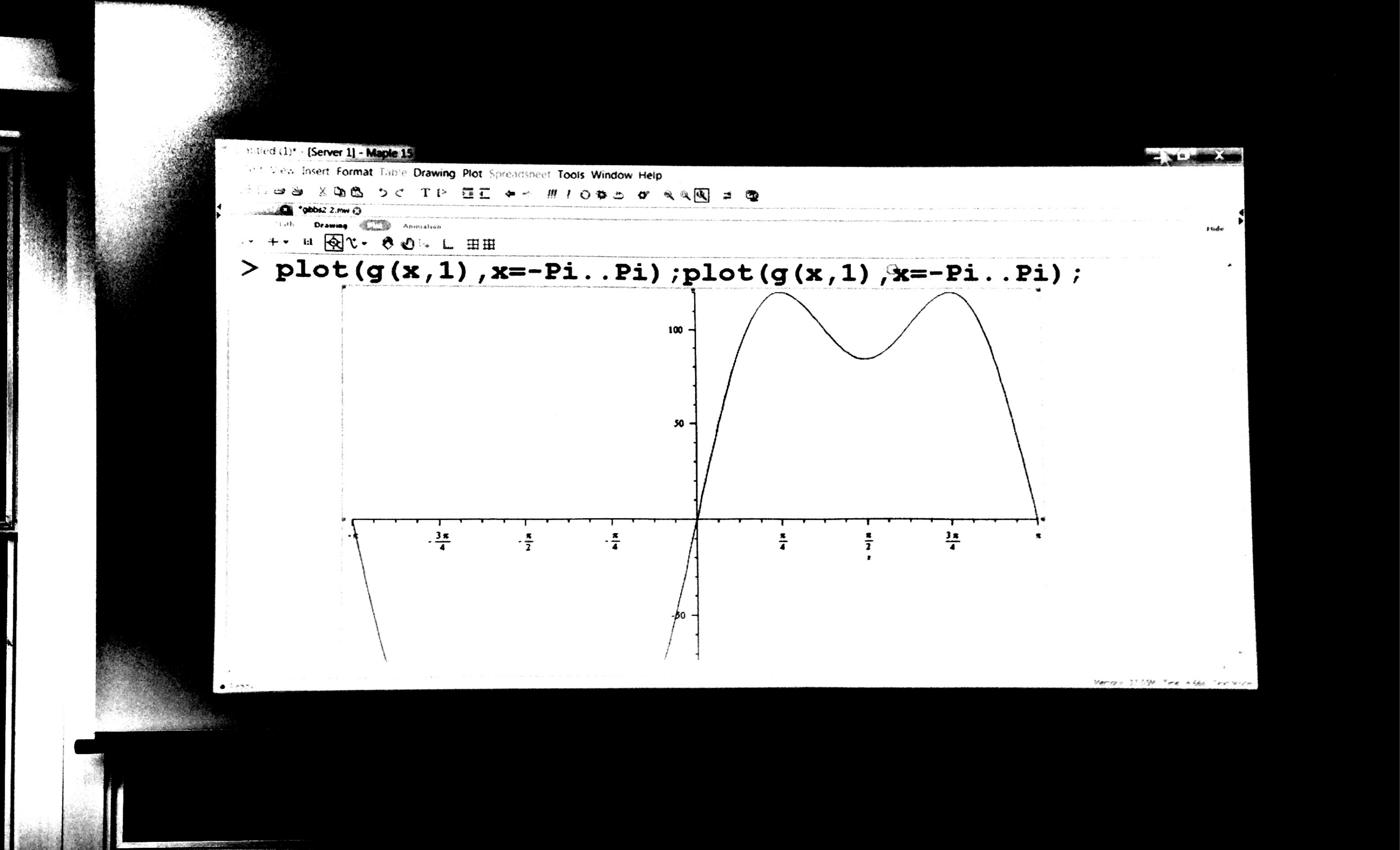

[Server 1] - Maple 15
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Drawing
Animation
> plot(g(x,1),x=-Pi..Pi);plot(g(x,1),x=-Pi..Pi);
100
50
-50
-3π/4
-π/2
-π/4
π/4
π/2
3π/4
π

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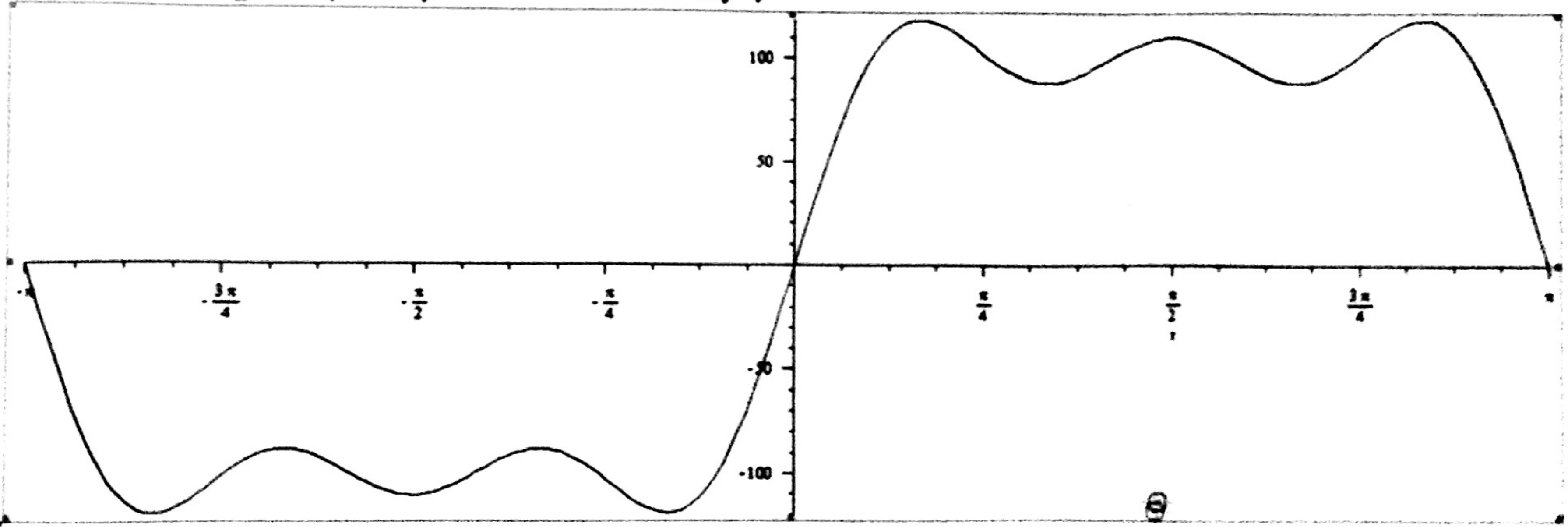
```
> plot(g(x,2),x=-Pi..Pi);
```




```
> plot(g(x,3),x=-Pi..Pi);
```

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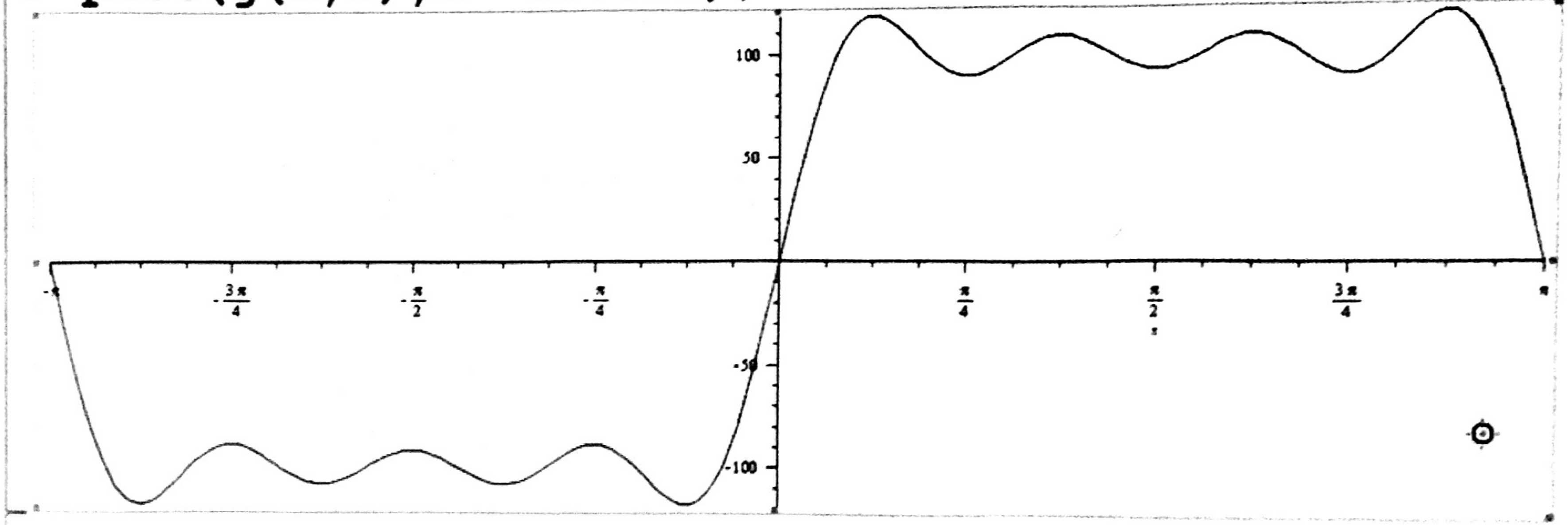
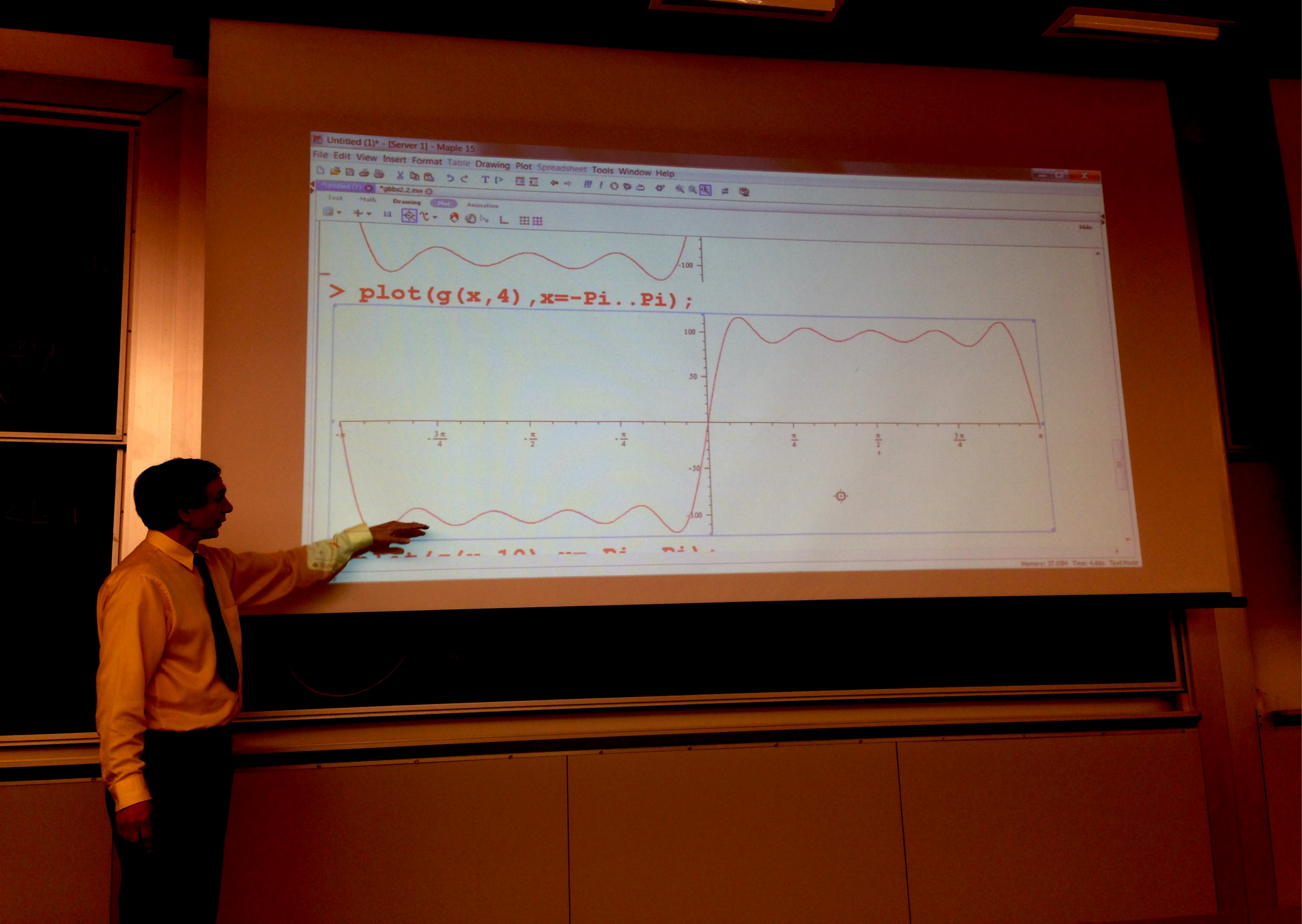
```
> plot(g(x,3),x=-Pi..Pi);
```




```
> plot(g(x,4),x=-Pi..Pi);
```

Ready



Untitled (1)* - [Server 1] - Maple 15
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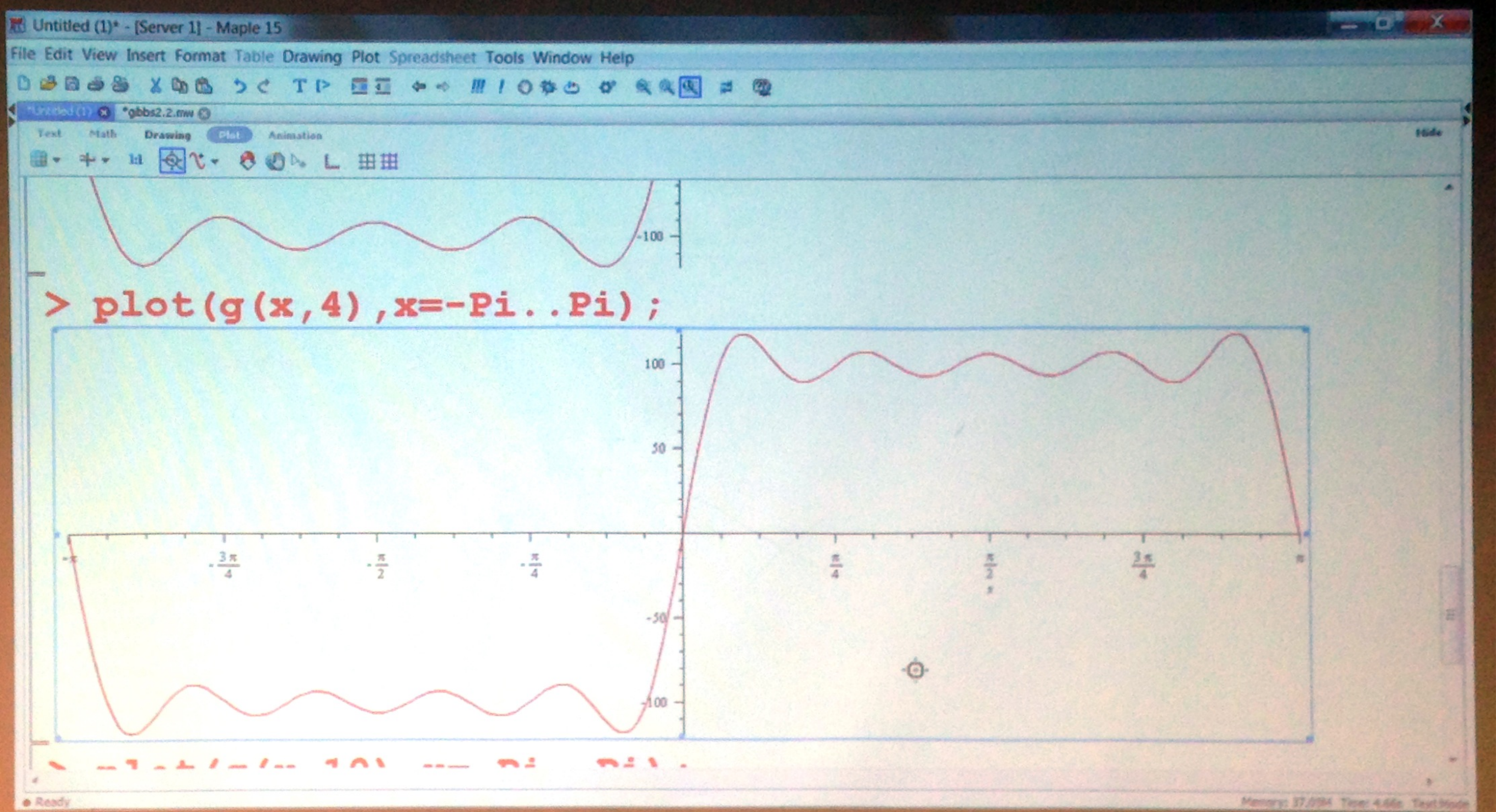
> plot(g(x,4),x=-Pi..Pi);



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-100
> plot(g(x,4),x=-Pi..Pi);
100
50
-50
-100
$-\pi$ $-\frac{3\pi}{4}$ $-\frac{\pi}{2}$ $-\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{2}$ $\frac{3\pi}{4}$ π
x
Ready

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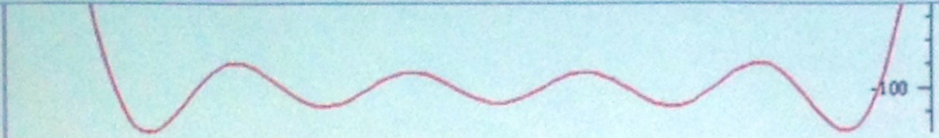
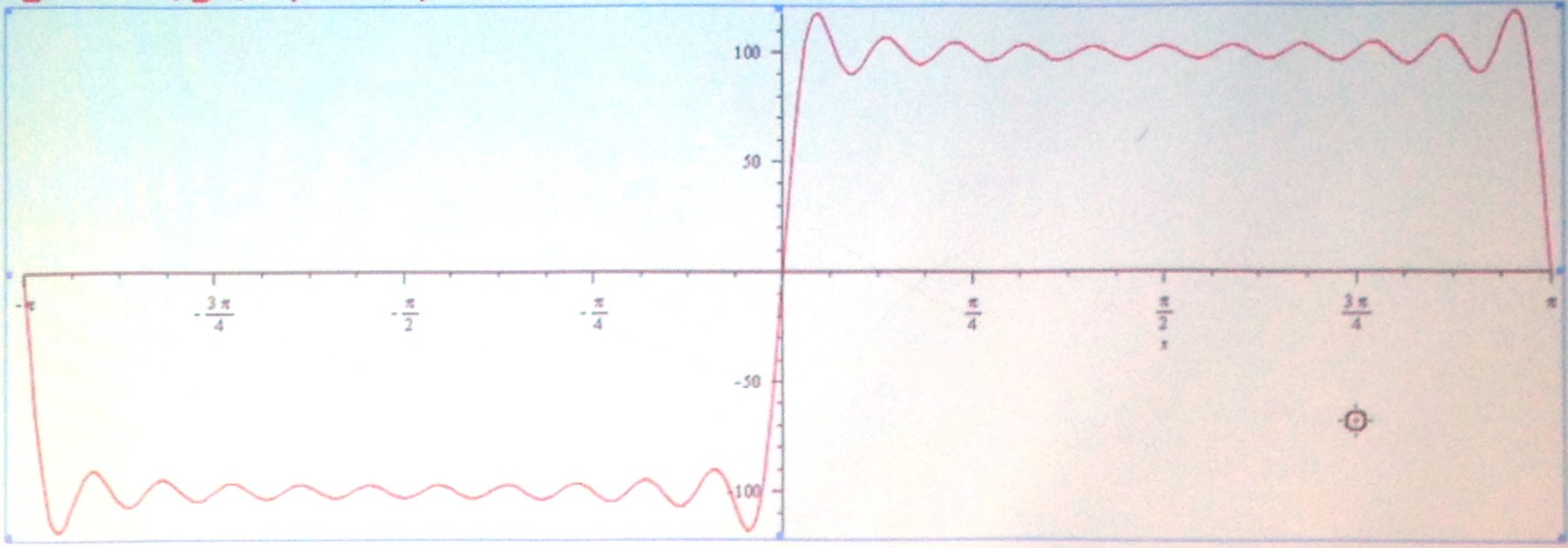
Text Math Drawing Plot Animation




```
> plot(g(x,10),x=-Pi..Pi);
```



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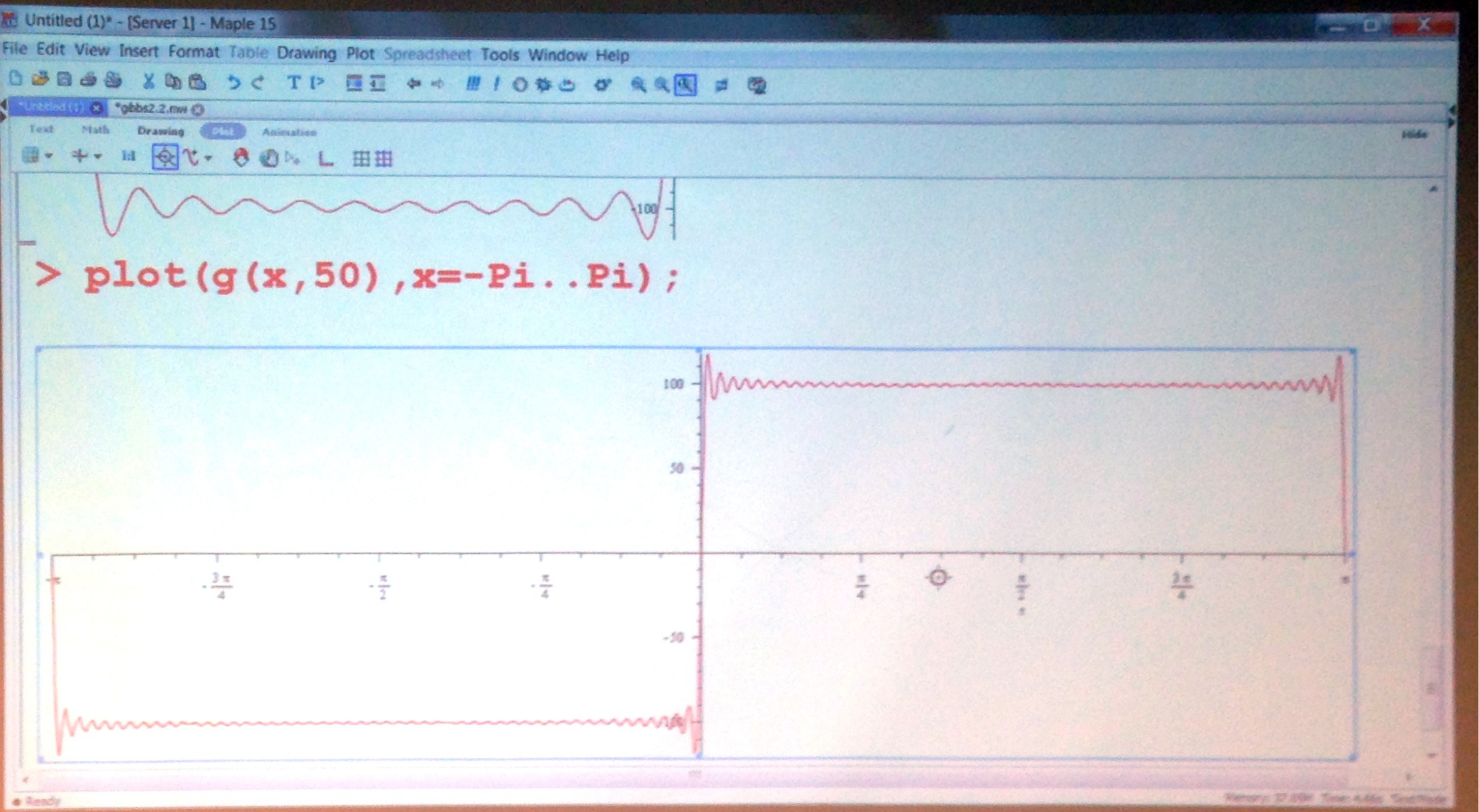
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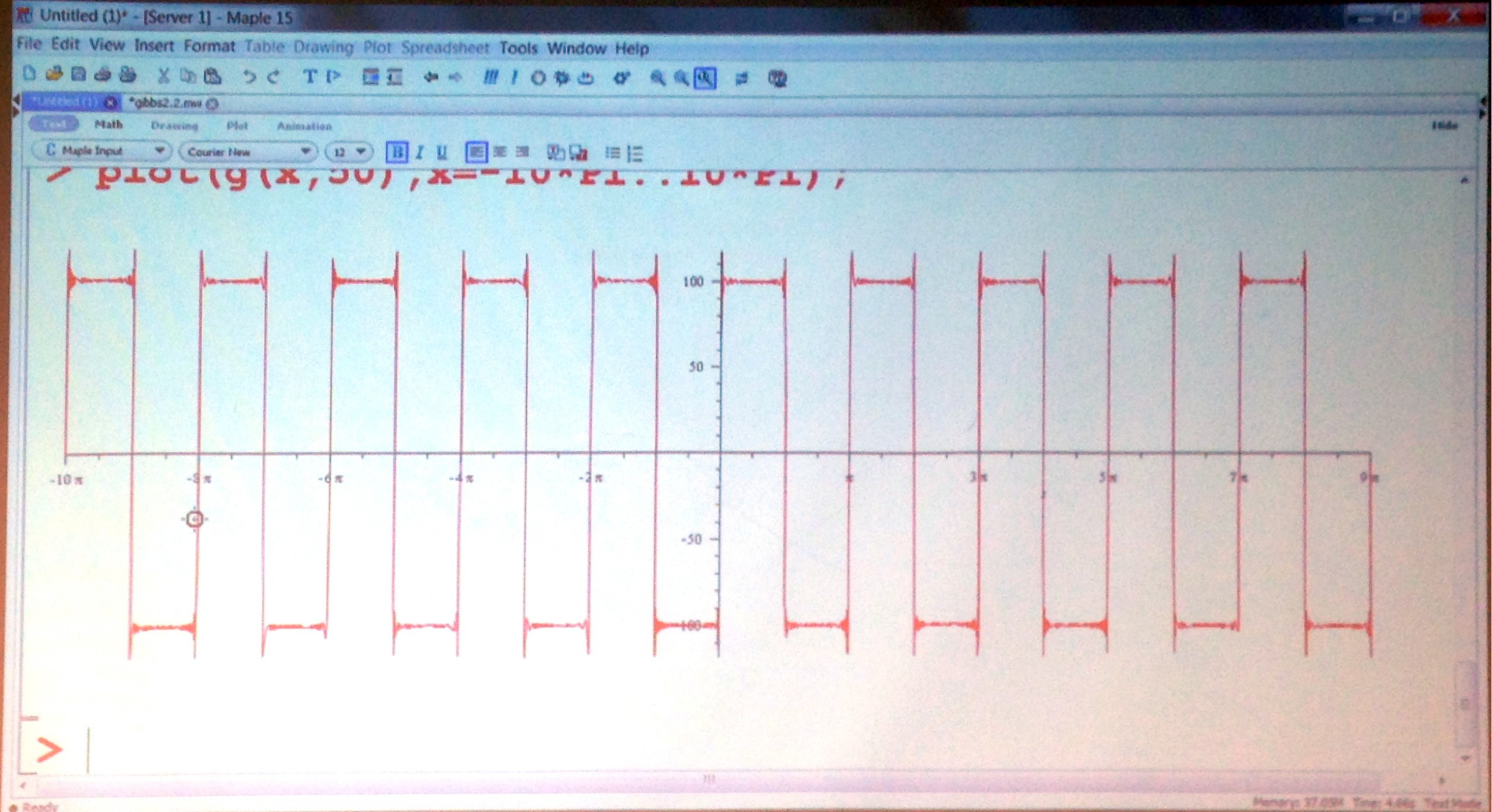
Hide
100
> plot(g(x,50),x=-Pi..Pi);
100
50
-50
-100
-π
-3π/4
-π/2
-π/4
π/4
π/2
3π/4
π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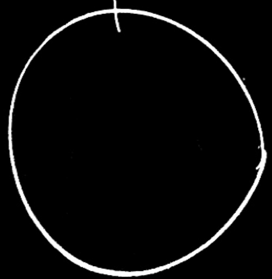
Text Math Drawing Plot Animation
Maple Input
Courier New
12
> plot(g(x,50),x=-10*Pi..10*Pi);
100
50
-50
-100
-10 π
-8 π
-6 π
-4 π
-2 π
π
3 π
5 π
7 π
9 π
>

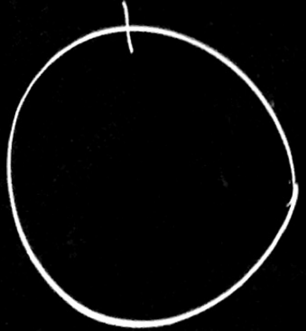
$$\begin{cases} u_t = u_{xx} & \text{(in polar coord)} \\ u(1,\theta) = f(\theta) \\ u \text{ periodic, period} = 2\pi \end{cases}$$



Eigenfunctions are sines + cosines

interval $= -\pi$ to π

$$\begin{cases} u_t = u_{xx} \quad \text{(in polar coord)} \\ u(1,\theta) = f(\theta) \\ u \text{ periodic, period} = 2\pi \end{cases}$$



↑ Eigenfunctions are sines + cosines

interval $= -\pi$ to π

← orthogonality

Orthogonality Trouble

$$\int_0^L h_1 h_2\, dx = 0 \qquad h_1 \neq h_2 \text{ in list } \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

$$0 < x < L$$

$$\int_{-L}^{L} h_1 h_2\, dx = 0 \qquad h_1 \neq h_2 \text{ in list of } 1, \text{sines, cosines}$$

$$-L < x < L$$

KEY Example

Is $\int_0^L \cos\left(\frac{2\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx = 0$?

No. Should be $-L<x<L$.

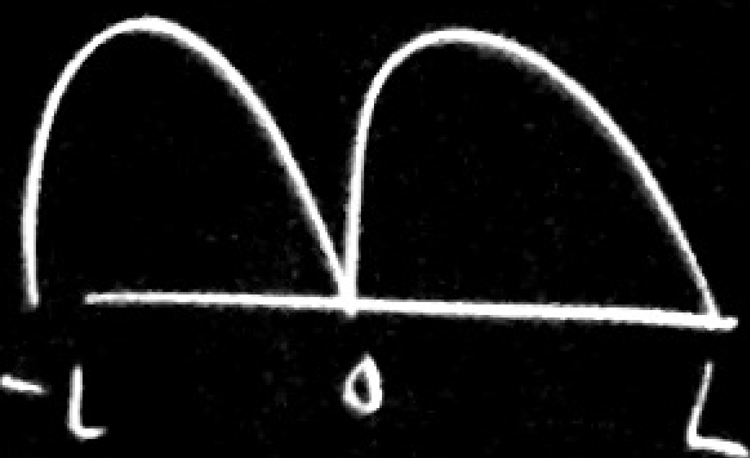
Theorem

Fourier series of f = Fourier cosine series of f_1

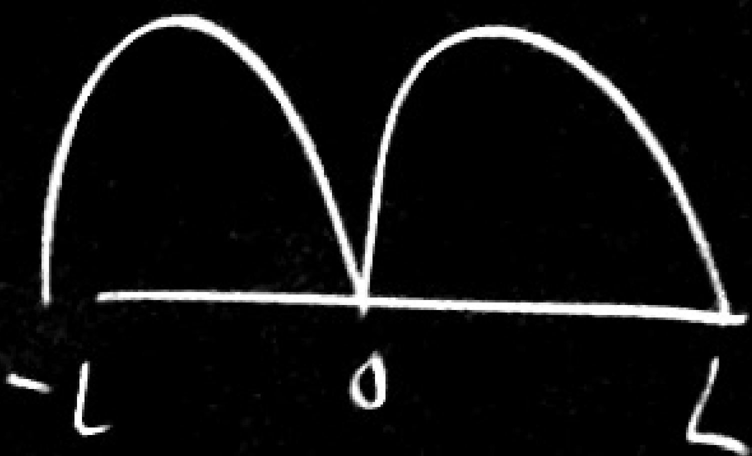
+ Fourier sine series of f_2

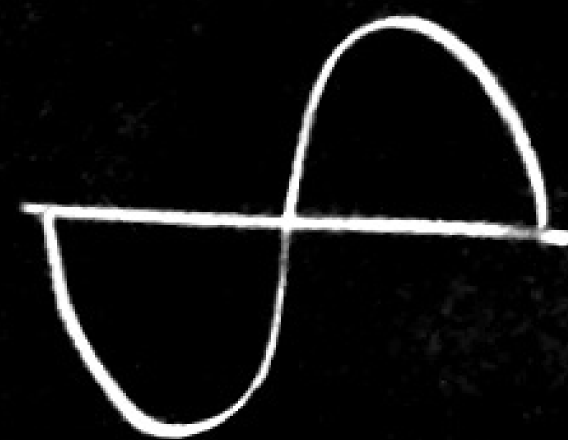
$f_1 = \begin{cases} f(x) & 0<x<L \\ f(-x) & -L<x<0 \end{cases}$ and $f_2(x) = \begin{cases} f(x) & 0<x<L \\ -f(-x) & -L<x<0 \end{cases}$

$$\int_{-L}^{L} (\text{even})\,dx = 2\int_{0}^{L} (\text{same})$$



$$\int_{-L}^{L} (\text{even})\,dx = 2\int_{0}^{L} (\text{same}) \quad , \quad \int_{-L}^{L} (\text{odd})\,dx = 0$$





KEY Example

Is $\int_0^L \cos\left(\frac{2\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx = 0$?

No. Should be $-L<x<L$.

Theorem

Fourier series of f = Fourier Cosine series of f_1

+ Fourier sine series of f_2

$f_1(x) = \frac{f(x)+f(-x)}{2}$, $f_2(x) = \frac{f(x)-f(-x)}{2}$

$-L \le x \le L$

f piecewise smooth May have jumps

<u>Thm</u> Fourier Series $F(x) = f(x)$ on $|x| \le L$ $\iff$ $f(x)$ is continuous (no jumps)
with F continuous (no jumps) and $f(-L) = f(L)$