Chapter H10/EPH 16

10.2 Heat equation, ∞ Domain

$u_t = k u_{xx}$, $t > 0$, $\underbrace{-\infty < x < \infty}_{\text{New}}$

$u(x,0) = f(x)$ $\quad -\infty < x < \infty$ initial Temp

$u(-\infty,t) = 0$, $u(\infty,t) = 0$ replaces $u(0,t) = 0, u(L,t) = 0$?

No. But usually correct

$u(x,t)$ bounded $\longleftarrow$ New

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10.2 Heat equation, ∞ Domain

$$\begin{cases} u_t = k u_{xx}, \quad t>0, \quad \underset{\text{New}}{\underline{-\infty < x < \infty}} \\ u(x,0) = f(x) \quad -\infty < x < \infty \quad \text{initial Temp} \\ u(-\infty,t)=0,\ u(\infty,t)=0 \quad \text{replace } u(0,t)=0,\ u(L,t)=0\,? \\ \text{No. But usually correct} \\ u(x,t) \text{ bounded} \longleftarrow \text{New} \end{cases}$$

Prod Sol $u = T(t)X(x)$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

split

$$\begin{cases} T' + k\lambda T = 0 \\ T(0) \neq 0 \end{cases} \qquad \begin{cases} X'' + \lambda X = 0 \\ X \text{ bounded} \end{cases}$$

ch 1 Discrete Spectrum

$$\omega = \frac{n\pi}{L}$$

ch 10 Continuous Spectrum

$$0 \le \lambda < \infty$$

Case $\lambda < 0$

$\lambda = -\omega^2$

$X = c_1 e^{\omega x} + c_2 e^{-\omega x}$

Not bounded

$\boxed{\text{No Sol}}$

Case $\lambda = 0$

$X = c_1 + c_2 x$

$c_2 = 0$

$\boxed{X = 1, \lambda = 0}$

Case $\lambda > 0$

$\lambda = \omega^2$

$\boxed{\begin{array}{l} X = c_1 \cos(\omega x) + c_2 \sin(\omega x) \\ \lambda = \omega^2 > 0 \end{array}}$

Complex form $X(x)$

$$X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

$$= c_1 \frac{e^{i\omega x} + e^{-i\omega x}}{2} + c_2 \frac{e^{i\omega x} - e^{-i\omega x}}{2i}$$

$$= d_1 e^{i\omega x} + d_2 e^{-i\omega x}$$

$$X = \text{constant} * e^{i\omega x} \qquad -\infty < \omega < \infty$$

Eigenpairs

$\lambda = \omega^2 \qquad 0 \le \omega < \infty$

$(\lambda, \cos \omega x), (\lambda, \sin \omega x)$

$\lambda = \omega^2, \ \omega = \omega \text{ or } -\omega$

$(\lambda, e^{i\omega x})$

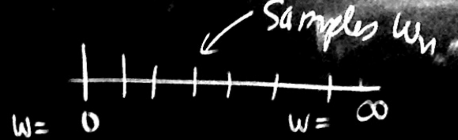
Superposition

Can't use Σ

add $T(t)X(x) = e^{-\omega^2 kt}\; A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)$

$0 \le \omega < \infty$

Superposition



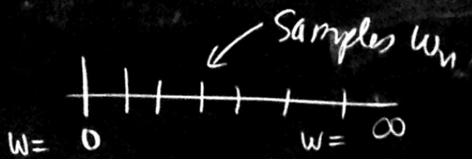
Can't use Σ

add $T(t)X(x) = e^{-\omega^2 kt}\left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)$

$0 \le \omega < \infty$

$u(x,t) = \sum$ prod sol's $= \sum_n e^{-\omega_n^2 kt}\left(A(\omega_n)\cos(\omega_n x) + B(\omega_n)\sin(\omega_n x)\right)$

Superposition



Can't use Σ

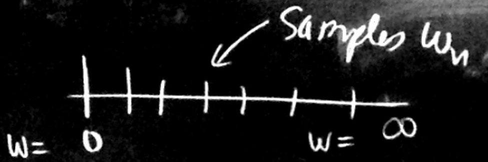
add $T(t)X(x) = e^{-\omega^2 kt}\left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)$

$0 \le \omega < \infty$

$$u(x,t) = \sum \text{prod sols} = \sum_n e^{-\omega_n^2 kt}\left(A(\omega_n)\cos(\omega_n x) + B(\omega_n)\sin(\omega_n x)\right)\Delta\omega_n \cong \int_{\omega=0}^{\omega=\infty} e^{-\omega^2 kt}\left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)d\omega$$

$$f(x) = u(x,0) = \int_0^\infty \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)d\omega$$

Superposition



Can't use $\sum$

add $T(t)X(x) = e^{-w^2 kt}\left(A(w)\cos(wx) + B(w)\sin(wx)\right)$

$0 \le w < \infty$

$$u(x,t) = \sum \text{prod sols} = \sum_n e^{-w_n^2 kt}\left(A(w_n)\cos(w_n x) + B(w_n)\sin(w_n x)\right)\Delta w_n \cong \int_{w=0}^{w=\infty} e^{-w^2 kt}\left(A(w)\cos(wx) + B(w)\sin(wx)\right)dw$$

$f(x) = u(x,0) = \int_0^\infty \left(A(w)\cos(wx) + B(w)\sin(wx)\right)dw$ Fourier Integral Theorem

$$T = e^{-\lambda k t}$$

$$X = e^{i \varpi x}$$

$$\varpi = \omega \text{ or } -\omega$$

$$\lambda = \omega^2 \geq 0$$

$$u(x,t) = \int_0^\infty \left(A\, \frac{e^{i\omega x} + e^{-i\omega x}}{2} + B\, \frac{e^{i\omega x} - e^{-i\omega x}}{2i} \right) d\omega$$

$$= \int_{0}^{\infty}{}_{\omega \geq 0} + \int_{0}^{\infty}{}_{\omega < 0} = \int_0^\infty C e^{i\omega x}\, d\omega + \int_0^\infty D e^{-i\omega x}\, d\omega$$

$$T = e^{-\lambda k t}$$

$$X = e^{i\Omega x}$$

$$\Omega = \omega \quad \Omega = -\omega$$

$$\lambda = \omega^2 \geq 0$$

$$u(x,t) = \int_0 \left(A \frac{e^{i\omega x} + e^{-i\omega x}}{2} + B \frac{e^{i\omega x} - e^{-i\omega x}}{2i} \right) d\omega$$

$$T = e^{-\lambda t}$$

$$X = e^{i\Omega x}$$

$$\Omega = \omega \quad \Omega = -\omega$$

$$\lambda = \omega^2 \geq 0$$

$$= \int_{\substack{0 \\ \omega \geq 0}}^{\infty} + \int_{\substack{0 \\ \omega < 0}}^{\infty} = \int_0^{\infty} C e^{i\omega x} d\omega + \int_0^{\infty} D e^{-i\omega x} d\omega$$

$$\begin{cases} \Omega = \omega \\ d\Omega = d\omega \end{cases} \qquad \begin{cases} \Omega = -\omega \\ d\Omega = -d\omega \end{cases}$$

$$= \int_0^{\infty} C e^{i\Omega x} d\Omega + \int_0^{-\infty} D(-\Omega) e^{i\Omega x} (-d\Omega) = \int_0^{\infty} C e^{i\Omega x} d\Omega + \int_{-\infty}^{0} D(-\Omega) e^{i\Omega x} d\Omega \equiv \int_{-\infty}^{\infty} E(\Omega) e^{i\Omega x} d\Omega$$

DEF: Fourier Transform pair

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

Why 2π?

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x} \quad \text{on } |x| \le L$$

Why 2π? assume $f(x)$ cont.

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x} \quad \text{on } |x| \leq L$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi}{L}x} dx$$

$$f(x) = \sum_{-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi}{L}x} dx \, e^{-i\frac{n\pi}{L}x}$$

Why 2π? assume $f(x)$ cont.

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{-i\frac{n\pi}{L}x} \quad \text{on } |x| \le L$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{i\frac{n\pi}{L}x}\,dx$$

If $L \to \infty$, then $\frac{n\pi}{L}$ get denser near 0

$$\omega_n = \frac{n\pi}{L} \qquad \omega_{n+1} - \omega_n = \frac{\pi}{L} = \Delta\omega_n$$

$$f(x) = \sum_{n=-\infty}^{n=\infty} \frac{1}{2L}\int_{-L}^{L} f(u)e^{i\frac{n\pi}{L}u}\,du\; e^{-i\frac{n\pi}{L}x} \cong \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)e^{i\omega_n u}\,du\right)e^{-i\omega_n x}\,\overset{\Delta\omega_n}{\cancel{\frac{\pi}{L}}} = \frac{1}{2\pi}\ \text{R.S. fn} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega u}\,du\,e^{-i\omega x}\,d\omega$$

Then $f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$ where $F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega u} du$

DEF.: Fourier Transform pair

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \, d\omega$$

Inverse Fourier Transform

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx$$

Fourier Transform of f(x)

Untitled (1)* - [Server 1] -

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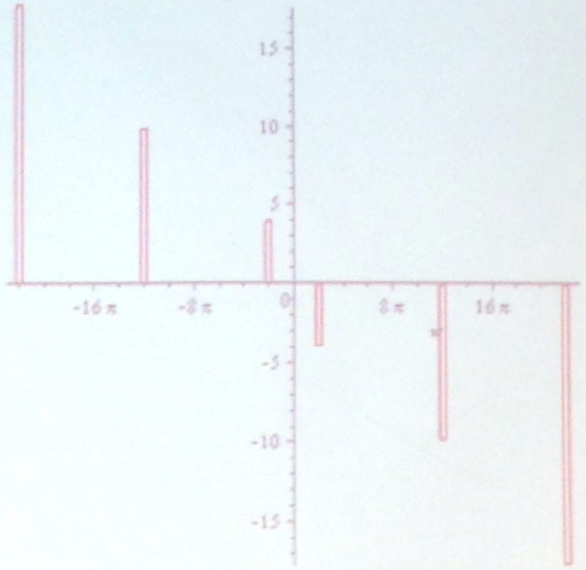
```
> # How to plot an impulse train.
> # The adjustment width h should be chosen for visual effect.
>
> f:=x->2*sin(2*Pi*x) + 5*sin(2*Pi*6*x) + 9*sin(2*Pi*11*x);
> Fw:=inttrans[fourier](f(x),x,w);h:='h';
> ApproxDirac:=x->(1/2/h)*(piecewise(x+h<0,0,1)-piecewise(x-h<0,0,1));
> F:=subs(Dirac=ApproxDirac,Fw);
> F:=F/I:
> h:=0.8:plot(F,w=-23*Pi..23*Pi);
>
>
```

Untitled (1)* - [Server 1] - Maple 15

```
> F:=F/I:
> h:=0.8:plot(F,w=-23*Pi..23*Pi);
>
>
```

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>
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$$f := x \rightarrow 2\sin(2\pi x) + 5\sin(12\pi x) + 9\sin(22\pi x) \qquad (1)$$

```
> Fw:=inttrans[fourier](f(x),x,w);h:='h';
```

$$Fw := \mathrm{I}\,\pi\,\big(9\,\mathrm{Dirac}(w+22\pi) + 2\,\mathrm{Dirac}(w+2\pi) - 2\,\mathrm{Dirac}(w-2\pi) + 5\,\mathrm{Dirac}(w+12\pi) - 5\,\mathrm{Dirac}(w-12\pi) - 9\,\mathrm{Dirac}(w-22\pi)\big)$$

$$h := h \qquad (2)$$

```
> ApproxDirac:=x->(1/2/h)*(piecewise(x+h<0,0,1)-piecewise(x-h<0,0,1));
```

$$ApproxDirac := x \rightarrow \frac{1}{2}\,\frac{piecewise(x+h<0,0,1) - piecewise(x-h<0,0,1)}{h} \qquad (3)$$

```
> F:=subs(Dirac=ApproxDirac,Fw);
```

$$F := \mathrm{I}\,\pi\,\big(9\,ApproxDirac(w+22\pi) + 2\,ApproxDirac(w+2\pi) - 2\,ApproxDirac(w-2\pi) + 5\,ApproxDirac(w+12\pi) - 5\,ApproxDirac(w-12\pi) - 9\,ApproxDirac(w-22\pi)\big) \qquad (4)$$

```
> F:=F/I:
> h:=0.8:plot(F,w=-23*Pi..23*Pi);
```

Ready