$\boxed{\text{BC}}$

$u=0$ at $x=0, x=L$

$X(x)T(t)=0$ at $x=0, x=L$

$T(t)\neq 0$ or $T(0)\neq 0$

$X(x)=0$ at $x=0, x=L$

$$\boxed{X(0)=0,\ X(L)=0}$$

$$\text{BVP}\begin{cases} X''+\lambda X=0 \\ X(0)=0,\ X(L)=0 \end{cases}$$

$\downarrow$

Solve by ODE Methods, 3 Cases

$\boxed{\text{THEOREM}}$ $X = C_2 \sin\left(\sqrt{\lambda}\,\dfrac{x}{L}\right)$

$$\begin{cases} T'+k\lambda T=0 \\ T(0)\neq 0 \end{cases}$$

IVP

$\downarrow$

$$T=\frac{\text{const}}{\text{integ factor}}$$

$$T=T(0)e^{-k\lambda t}$$

prod Sol $X(x)T(t)$

<u>theorem</u> $T(t)=T(0)e^{-k\lambda t}$

BVP has a Sol $X(x)\neq 0$ if and only if $\lambda>0$, $\sqrt{\lambda}L=n\pi$ for $n=1,2,3,\ldots$ and

$$X(x)=c_2\sin(\sqrt{\lambda}X/L)$$

Kod Problem

$\boxed{\text{PDE}}$ $u_t=k\,u_{xx}$

$\boxed{\text{BC}}$ $u=0$ at $x=0, x=L$

$\boxed{\text{IC}}$ $u(x,0)=f(x)$

Char eq : $r^2 + \lambda = 0$

Discr $= b^2 - 4ac$

$= 0 - 4 \cdot 1 \cdot \lambda$

$= -4\lambda$.

3 Cases

(1) real roots $\lambda < 0$

(2) double root $\lambda = 0$

(3) Complex roots $\lambda > 0$

Char eq: $r^2 + \lambda = 0$ Discr $= b^2 - 4ac$

$= 0 - 4 \cdot 1 \cdot \lambda$

$= -4\lambda$

3 Cases

(1) Real Roots $\lambda < 0$ $X = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

(2) double root $\lambda = 0$ $X = (C_1 + C_2 x) e^{r_1 x}$

(3) Complex roots $\lambda > 0$ $X = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

Char eq: $r^2+\lambda=0$

Discr $= b^2-4ac$

$= 0-4\cdot1\cdot\lambda$

$= -4\lambda$

3 Cases

(1) real roots $\lambda<0$ $\quad X = C_1e^{r_1x}+C_2e^{r_2x}$

(2) double root $\lambda=0$ $\quad X=(C_1+C_2x)e^{r_1x}$

Monotonic

(3) complex roots $\lambda>0$ $\quad X = C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

Case 3 $\quad X=0$ at $x=0$, $x=L$

Means

$x=0$: $\quad 0 = C_1\overset{1}{\cancel{\cos(0)}}+C_2\overset{0}{\cancel{\sin(0)}}$

$x=L$: $\quad 0 = C_2\sin(\sqrt{\lambda}L)$

$\sqrt{\lambda}L = n\pi$