4.9 Orthogonal Bases and the Gram-Schmidt Algorithm

In Section 4.8 we discussed the problem of finding the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{b}$ into the subspace V of $\mathbf{R}^m$. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis for V, and the $m \times n$ matrix A has these basis vectors as its column vectors, then the orthogonal projection $\mathbf{p}$ is given by

$$\mathbf{p} = A\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is the (unique) solution of the normal system

$$A^T A\mathbf{x} = A^T\mathbf{b}. \tag{2}$$

The formula for $\mathbf{p}$ takes an especially simple and attractive form when the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are mutually orthogonal.

DEFINITION Orthogonal Basis

An **orthogonal basis** for the subspace V of $\mathbf{R}^m$ is a basis consisting of vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_n$ that are mutually orthogonal, so that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$. If in addition these basis vectors are unit vectors, so that $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, \ldots, n$, then the orthogonal basis is called an **orthonormal basis.**

Example 1 The vectors

$$\mathbf{v}_1 = (1, 1, 0), \quad \mathbf{v}_2 = (1, -1, 2), \quad \mathbf{v}_3 = (-1, 1, 1)$$

form an orthogonal basis for $\mathbf{R}^3$. We can "normalize" this orthogonal basis by dividing each basis vector by its length: If

$$\mathbf{w}_i = \frac{\mathbf{v}_i}{|\mathbf{v}_i|} \quad (i = 1, 2, 3),$$

then the vectors

$$\mathbf{w}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \mathbf{w}_2 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \mathbf{w}_3 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

form an orthonormal basis for $\mathbf{R}^3$. ■

Now suppose that the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of the $m \times n$ matrix A form an *orthogonal* basis for the subspace V of $\mathbf{R}^m$. Then

$$A^T A = [\mathbf{v}_i \cdot \mathbf{v}_j] = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{v}_n \cdot \mathbf{v}_n \end{bmatrix}. \tag{3}$$

Thus the coefficient matrix in the normal system in (2) is a diagonal matrix, so the normal equations simplify to

$$\begin{aligned} (\mathbf{v}_1 \cdot \mathbf{v}_1)x_1 &= \mathbf{v}_1 \cdot \mathbf{b} \\ (\mathbf{v}_2 \cdot \mathbf{v}_2)x_2 &= \mathbf{v}_2 \cdot \mathbf{b} \\ &\vdots \\ (\mathbf{v}_n \cdot \mathbf{v}_n)\mathbf{x}_n &= \mathbf{v}_n \cdot \mathbf{b} \end{aligned} \tag{4}$$

Consequently the solution $\mathbf{x} = (x_1 x_2, \ldots, x_n)$ of the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is given by

$$x_i = \frac{\mathbf{v}_i \cdot \mathbf{b}}{\mathbf{v}_i \cdot \mathbf{v}_i} \quad (i = 1, 2, \ldots, n). \tag{5}$$

When we substitute this solution in Equation (1),

$$\mathbf{p} = A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

we get the following result.

THEOREM 1 Projection into an Orthogonal Basis

Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form an orthogonal basis for the subspace V of $\mathbf{R}^n$. Then the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{b}$ into V is given by

$$\mathbf{p} = \frac{\mathbf{v}_1 \cdot \mathbf{b}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{b}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \cdots + \frac{\mathbf{v}_n \cdot \mathbf{b}}{\mathbf{v}_n \cdot \mathbf{v}_n}\mathbf{v}_n. \tag{6}$$

Note that if the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is orthonormal rather than merely orthogonal, then the formula for the orthogonal projection $\mathbf{p}$ simplifies still further:

$$\mathbf{p} = (\mathbf{v}_1 \cdot \mathbf{b})\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{b})\mathbf{v}_2 + \cdots + (\mathbf{v}_n \cdot \mathbf{b})\mathbf{v}_n.$$

If $n = 1$ and $\mathbf{v} = \mathbf{v}_1$, then the formula in (6) reduces to the formula

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} \tag{7}$$

for the **component of b parallel to v**. The **component of b orthogonal to v** is then

$$\mathbf{q} = \mathbf{b} - \mathbf{p} = \mathbf{b} - \frac{\mathbf{v} \cdot \mathbf{b}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}. \tag{8}$$

Theorem 1 may also be stated without symbolism: *The orthogonal projection* **p** *of the vector* **b** *into the subspace V is the sum of the components of* **b** *parallel to the orthogonal basis vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *for V.*

Example 2 The vectors

$$\mathbf{v}_1 = (1, 1, 0, 1), \quad \mathbf{v}_2 = (1, -2, 3, 1), \quad \mathbf{v}_3 = (-4, 3, 3, 1)$$

are mutually orthogonal, and hence form an orthogonal basis for a 3-dimensional subspace V of $\mathbf{R}^4$. To find the orthogonal projection **p** of $\mathbf{b} = (0, 7, 0, 7)$ into V we use the formula in (6) and get

$$\begin{aligned}\mathbf{p} &= \frac{\mathbf{v}_1 \cdot \mathbf{b}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{b}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{v}_3 \cdot \mathbf{b}}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 \\ &= \frac{14}{3}(1, 1, 0, 1) + \frac{-7}{15}(1, -2, 3, 1) + \frac{28}{35}(-4, 3, 3, 1);\end{aligned}$$

therefore

$$\mathbf{p} = (1, 8, 1, 5).$$

The component of the vector **b** orthogonal to the subspace V is

$$\mathbf{q} = \mathbf{b} - \mathbf{p} = (0, 7, 0, 7) - (1, 8, 1, 5) = (-1, -1, -1, 2).$$

Example 2 illustrates how useful orthogonal bases can be for computational purposes. There is a standard process that is used to transform a given *linearly independent* set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{R}^m$ into a *mutually orthogonal* set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that span the same subspace of $\mathbf{R}^m$. We begin with

$$\mathbf{u}_1 = \mathbf{v}_1. \tag{9}$$

To get the second vector $\mathbf{u}_2$, we subtract from $\mathbf{v}_2$ its component parallel to $\mathbf{u}_1$. That is,

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \tag{10}$$

is the component of $\mathbf{v}_2$ orthogonal to $\mathbf{u}_1$. At this point it is clear that $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthogonal basis for the 2-dimensional subspace V_2 spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. To get the third vector $\mathbf{u}_3$, we subtract from $\mathbf{v}_3$ its components parallel to $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \tag{11}$$

is the component of $\mathbf{v}_3$ orthogonal to the subspace V_2 spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Having defined $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_k$ in this manner, we take $\mathbf{u}_{k+1}$ to be the component of $\mathbf{v}_{k+1}$ orthogonal to the subspace V_k spanned by $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_k$. This process for constructing the mutually orthogonal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is summarized in the following algorithm.

ALGORITHM Gram–Schmidt Orthogonalization

To replace the linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ one by one with mutually orthogonal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that span the same subspace of $\mathbf{R}^m$, begin with

$$\mathbf{u}_1 = \mathbf{v}_1. \tag{12}$$

For $k = 1, 2, \ldots, \mathbf{n} - 1$ in turn, take

$$\begin{aligned} \mathbf{u}_{k+1} = \mathbf{v}_{k+1} &- \frac{\mathbf{u}_1 \cdot \mathbf{v}_{k+1}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\ &- \frac{\mathbf{u}_2 \cdot \mathbf{v}_{k+1}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k+1}}{\mathbf{u}_k \cdot \mathbf{u}_k}\mathbf{u}_k. \end{aligned} \tag{13}$$

The formula in (13) means that we get $\mathbf{u}_{k+1}$ from $\mathbf{v}_{k+1}$ by subtracting from $\mathbf{v}_{k+1}$ its components parallel to the mutually orthogonal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ previously constructed, and hence $\mathbf{u}_{k+1}$ is orthogonal to each of the first k vectors. If we assume inductively that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ span the same subspace as the original k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, then it also follows from (13) that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k+1}$ span the same subspace as the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}$. If we begin with a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for the subspace V of $\mathbf{R}^n$, the final result of carrying out the Gram-Schmidt algorithm is therefore an *orthogonal* basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ for V.

THEOREM 2 Existence of Orthogonal Bases

Every nonzero subspace of $\mathbf{R}^m$ has an orthogonal basis.

As a final step, one can convert the orthogonal basis to an *orthonormal* basis by dividing each of the orthogonal basis vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ by its length.

In numerical applications of the Gram-Schmidt algorithm, the calculations involved often can be simplified by multiplying each vector $\mathbf{u}_k$ (as it is found) by an appropriate scalar factor to eliminate unwanted fractions. We do this in the two examples that follow.

Example 3 To apply the Gram-Schmidt algorithm beginning with the basis

$$\mathbf{v}_1 = (3, 1, 1), \quad \mathbf{v}_2 = (1, 3, 1), \quad \mathbf{v}_3 = (1, 1, 3)$$

for $\mathbf{R}^3$, we first take

$$\mathbf{u}_1 = \mathbf{v}_1 = (3, 1, 1).$$

Then

$$\begin{aligned} \mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 &= (1, 3, 1) - \frac{7}{11}(3, 1, 1) \\ &= \left(-\frac{10}{11}, \frac{26}{11}, \frac{4}{11}\right) = \frac{2}{11}(-5, 13, 2). \end{aligned}$$

We delete the scalar factor $\frac{2}{11}$ (or multiply by $\frac{11}{2}$) and instead use $\mathbf{u}_2 = (-5, 13, 2)$. Next,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$
$$= (1, 1, 3) - \frac{7}{11}(3, 1, 1) - \frac{14}{198}(-5, 13, 2)$$
$$= \left(-\frac{55}{99}, -\frac{55}{99}, \frac{220}{99}\right) = -\frac{5}{9}(1, 1, -4).$$

We delete the scalar factor $-\frac{5}{9}$ and thereby choose $\mathbf{u}_3 = (1, 1, -4)$. Thus our final result is the orthogonal basis

$$\mathbf{u}_1 = (3, 1, 1), \quad \mathbf{u}_2 = (-5, 13, 2), \quad \mathbf{u}_3 = (1, 1, -4)]$$

for $\mathbf{R}^3$.

Example 4 Find an orthogonal basis for the subspace V of $\mathbf{R}^4$ that has basis vectors

$$\mathbf{v}_1 = (2, 1, 2, 1), \quad \mathbf{v}_2 = (2, 2, 1, 0), \quad \mathbf{v}_3 = (1, 2, 1, 0).$$

Solution We begin with

$$\mathbf{u}_1 = \mathbf{v}_1 = (2, 1, 2, 1).$$

Then

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = (2, 2, 1, 0) - \frac{8}{10}(2, 1, 2, 1)$$
$$= \left(\frac{2}{5}, \frac{6}{5}, -\frac{3}{5}, -\frac{4}{5}\right) = \frac{1}{5}(2, 6, -3, -4).$$

We delete the scalar factor $\frac{1}{5}$ and instead take $\mathbf{u}_2 = (2, 6, -3, -4)$. Next,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$
$$= (1, 2, 1, 0) - \frac{6}{10}(2, 1, 2, 1) - \frac{11}{65}(2, 6, -3, -4)$$
$$= \left(-\frac{70}{130}, \frac{50}{130}, \frac{40}{130}, \frac{10}{130}\right) = \frac{1}{13}(-7, 5, 4, 1).$$

We delete the scalar factor $\frac{1}{13}$ and instead take $\mathbf{u}_3 = (-7, 5, 4, 1)$. Thus our orthogonal basis for the subspace V consists of the vectors

$$\mathbf{u}_1 = (2, 1, 2, 1), \quad \mathbf{u}_2 = (2, 6, -3, -4), \quad \mathbf{u}_3 = (-7, 5, 4, 1).$$