<center>
Math 2250 Maple Lab 10
Brine Tanks. Glucose-Insulin Model.
S2014
</center>

Maple Project 10 has four problems L10-1, L10-2, L10-3, L10-4.

Submitted work. Please submit one stapled package. Some handwritten work is expected, which should display the steps shown in examples below. Maple code is expected to parallel the code given in the examples below. The final pages are appendices made by printing one or more maple work sheets.

References: Code in `maple` appears in `2250mapleL10-S2014.txt` at URL `http://www.math.utah.edu/~gustafso/S2014/`. This document: `2250mapleL10-S2014.pdf`.

Linear Cascade

Cascading brine tanks are modeled by a triangular system of differential equations. We will consider the three-tank model,

$$
\begin{aligned}
x_1'(t) &= -0.5x_1(t), \\
x_2'(t) &= 0.5x_1(t) - 0.25x_2(t), \\
x_3'(t) &= 0.25x_2(t) - 0.2x_3(t),
\end{aligned}
\tag{1}
$$

with initial conditions

$$x_1(0) = 15, \quad x_2(0) = 0, \quad x_3(0) = 0.$$

Glucose-Insulin Simplified Model

Let $G(t)$ be the excess glucose concentration in a person's blood at time t, and let $H(t)$ be the excess insulin concentration in the blood at time t. Concentration units are milligrams per deciliter (mg/dl). We consider the example

$$
\begin{aligned}
G'(t) &= -0.1G(t) - 0.4H(t), \\
H'(t) &= 0.1G(t) - 0.1H(t),
\end{aligned}
\tag{2}
$$

with initial data

$$G(0) = 100, \quad H(0) = 0.$$

Overview: Cells rely on glucose as their main energy supply. Glucose is a by-product of the food we eat. After digestion of the food, glucose is secreted into the bloodstream for transport to the cells. Simple, but glucose cannot enter the cells directly, because cells require insulin from the pancreas for cell uptake. The amount of insulin secreted by the pancreas is regulated by a feedback loop that measures the current blood glucose level. There is a normal ideal steady-state glucose concentration (70-120 mg/dl) in the blood maintained by the feedback loop. Too low (glucose less than 40 mg/dl) is the disease hypoglycemia and too high (glucose greater than 210 mg/dl) is the disease hyperglycemia [diabetes]. If glucose is at steady-state (homeostatic), then the pancreas secretion rate is about 500 mg/min, in order to maintain homeostasis. Briefly, the steady-state corresponds to zero excess concentrations, $G = H = 0$.

As food is digested, glucose levels rise and the feedback loop causes the pancreas to secret excess insulin. Glucose levels then fall, due to cell uptake. The feedback loop then reacts to lower the secretion of insulin.

The dynamical model for G and H captures the essentials of insulin secretion and glucose uptake, after a big meal, when no additional glucose is being added to the system. No derivation nor justification of this model is reproduced here. Suffice it to say, that the model is a rough approximation to the physical situation, obtained by linearization of a suitable nonlinear dynamical model.

Problem L10-1. (**Brine Cascade Modeling Details**)

Assume in the brine tank cascade model tank volumes $V_1 = 20$, $V_2 = 40$, $V_3 = 50$ and equal rate in and out of each tank of $r = 10$ liters per minute. The symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ are the amounts of salt in kilograms in each tank at time t. Complete these steps:

- Derive equations (1) using compartment analysis and the flow rate equation

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

  Use textbook page references and equations to decrease the effort of the derivation.

- Explain how the numbers 0.5, 0.25 and 0.2 relate to the tank volume data and the rate $r = 10$.

- Explain the physical meaning of the initial data $x_1(0) = 15$, $\quad x_2(0) = 0$, $\quad x_3(0) = 0$.

Problem L10-2. (**Brine Cascade Solution Details**)

Solve system (1) to obtain the answer

$$
\begin{array}{rll}
x_1(t) &=& 15e^{-t/2}, \\
x_2(t) &=& -30e^{-t/2} + 30e^{-t/4}, \\
x_3(t) &=& 25e^{-t/2} - 150e^{-t/4} + 125e^{-t/5},
\end{array}
$$

by completing these steps:

- Solve by the linear integrating factor method, with paper and pencil, applied to each differential equation in the cascade.

- Solve with Maple's `dsolve()`, or its equivalent in any computer algebra system.

- Graph the three component solutions on $0 \le t \le 10$ in a single graphic.

Problem L10-3. (**Glucose-Insulin Model Solution Details**)

Solve system (2) to obtain the answer
$$
\begin{array}{rll}
G(t) &=& 100e^{-t/10}\cos(t/5), \\
H(t) &=& 50e^{-t/10}\sin(t/5),
\end{array}
$$

by completing these steps:

- Solve with paper and pencil. Use the Cayley-Hamilton method for $u' = Au$. This method says that G, H are linear combinations of the two atoms obtained from the roots r of the characteristic equation $\det(A - rI) = 0$.

- Solve with Maple's `dsolve()`, or its equivalent in any computer algebra system.

- Graph the two component solutions G, H on $0 \le t \le 10$.

Problem L10-4. (**Glucose-Insulin Model Phase Plot**)

Make a phase portrait of G, H which shows that the trajectory limits to equilibrium $G = H = 0$ as $t \to \infty$. The plot should display at least four different starting values for G, H, one of which is $G(0) = 100$, $H(0) = 0$.

Staple this page on top of the maple work sheets. **Start paper and pencil solutions on a separate page.**
Examples on the next page . . .

Example 1. Solve the linear cascade $x_1' = -x_1$, $x_2' = x_1 - 2x_2$ with paper and pencil, then check the answer using Maple's `dsolve()`..

Solution: The first equation has the form $y' = -y$ with growth-decay solution $y = ce^{-x}$. Swapping symbols, $x_1(t) = c_1e^{-t}$. Substitute the answer into $x_2' = x_1 - 2x_2$, giving the linear equation $x_2' = -2x_2 + c_1e^{-t}$. This is a linear equation $y' + 2y = c_1e^{-x}$, solved by the linear integrating factor method, as follows:

$$y' + 2y = c_1e^{-x}$$
$$\frac{(yW)'}{W} = c_1e^{-x}, \text{ where } W = e^{\int 2dx} = e^{2x}$$
$$(yW)' = c_1We^{-x}, \text{ by cross-multiply}$$
$$yW = c_1 \int e^{2x}e^{-x}dx + c_2 = c_1e^{x} + c_2, \text{ by integration}$$
$$y = c_1e^{-x} + c_2e^{-2x}, \text{ divide by } W - e^{2x}$$

Then a swap of variable names gives the answer $x_2 = c_1e^{-t} + c_2e^{-2t}$ and finally the cascade has been solved:

$$x_1(t) = c_1e^{-t}, \quad x_2(t) = c_1e^{-t} + c_2e^{-2t}.$$

The maple answer check is done as follows:

```
de:=diff(x1(t),t)=-x1(t),diff(x2(t),t)=x1(t)-2*x2(t);
dsolve({de},{x1(t),x2(t)});
```

The maple output is $\left\{ x2\,(t) = \_C2\,e^{-t} + e^{-2\,t}\_C1, x1\,(t) = \_C2\,e^{-t} \right\}$, which is equivalent to the answer obtained by hand, via a re-naming of constants. The curious ordering of the factors and the underline that starts a constant name are features of maple.

Example 2. Apply the Cayley-Hamilton method to $u' = Au$, for $A = \begin{pmatrix} -1 & -9 \\ 1 & -1 \end{pmatrix}$.

Solution: Let vector u have components x_1, x_2. The differential equation $u' = Au$ in scalar form is the system

$$\begin{array}{rcrcr} x_1' & = & -x_1 & - & 9x_2, \\ x_2' & = & x_1 & - & x_2. \end{array}$$

The characteristic polynomial of A is the determinant of the matrix $A - rI$, where I is the 2×2 identity matrix. Expansion of the determinant gives the polynomial $r^2 + 2r + 10$. The characteristic equation is $r^2 + 2r + 10 = 0$, which has complex conjugate roots $-1 \pm 3i$. The atoms for these two roots are $e^{-t}\cos 3t$ and $e^{-t}\sin 3t$. The Cayley-Hamilton method implies that for some constants c_1, c_2,

$$x_1(t) = c_1e^{-t}\cos 3t + c_2e^{-t}\sin 3t.$$

Use the first differential equation $x_1' = -x_1 + 9x_2$ to solve for

$$\begin{aligned} x_2 & = & \frac{1}{9}(x_1' + x_1) \\ & = & \frac{1}{9}(-x_1 - 3c_1e^{-t}\sin 3t + 3c_2e^{-t}\cos 3t) \\ & = & (c_2/3 - c_1/9)e^{-t}\cos 3t + (-c_1/3 - c_2/9)e^{-t}\sin 3t. \end{aligned}$$

Example 3. Solve with Maple's `dsolve()` the equation $u' = Au$, for $A = \begin{pmatrix} -1 & -9 \\ 1 & -1 \end{pmatrix}$, $u(0) = \begin{pmatrix} 300 \\ 0 \end{pmatrix}$.

Solution: Let vector u have components x_1, x_2, written in maple as `x1, x2`. Using the scalar form of system $u' = Au$, from the previous example, the maple code is:

```
de:=diff(x1(t),t)=-x1(t)-9*x2(t),diff(x2(t),t)=x1(t)-x2(t);
ic:=x1(0)=300,x2(0)=0;
dsolve({de,ic},{x1(t),x2(t)});
```

The answer reported by maple is

$$\left\{ x1\,(t) = 300\,e^{-t}\cos\,(3\,t), x2\,(t) = 100\,e^{-t}\sin\,(3\,t) \right\}$$

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Example 4. Make a direction field plot and a phase portrait for the system $u' = Au$, where $A = \begin{pmatrix} -1 & -9 \\ 1 & -1 \end{pmatrix}$. Include one trajectory for $u(0) = \begin{pmatrix} 300 \\ 0 \end{pmatrix}$.

Solution: The work will use a basic package available in all maple versions. Starting with maple 10, there are other alternatives available in the dynamical systems package, perhaps slightly easier to use, especially for experiments.

To make a direction field plot:

```
de:=diff(x1(t),t)=-x1(t)-9*x2(t),diff(x2(t),t)=x1(t)-x2(t);
window:=x1=-400..400,x2=-100..100; tval:=-10..10:
with(DEtools):
DEplot({de},{x1(t),x2(t)},tval,window);
```
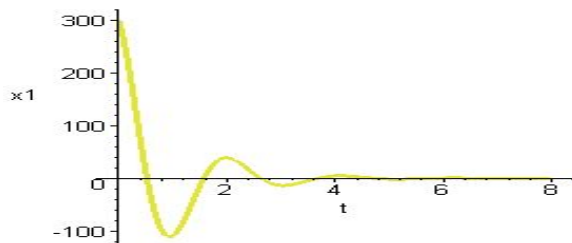
To make a phase portrait with threaded curves started at dots $(300, 0)$, $(200, 0)$, $(100, 50)$, $(50, 20)$:

```
de:=diff(x1(t),t)=-x1(t)-9*x2(t),diff(x2(t),t)=x1(t)-x2(t);
dot:=(a,b)->[x1(0)=a,x2(0)=b];
ic:=[dot(300,0),dot(200,0),dot(100,50),dot(50,20)];
window:=x1=-400..400,x2=-100..100; tval:=-10..10:
opts:=stepsize=0.05;
with(DEtools):
DEplot({de},{x1(t),x2(t)},tval,window,ic,opts);
```
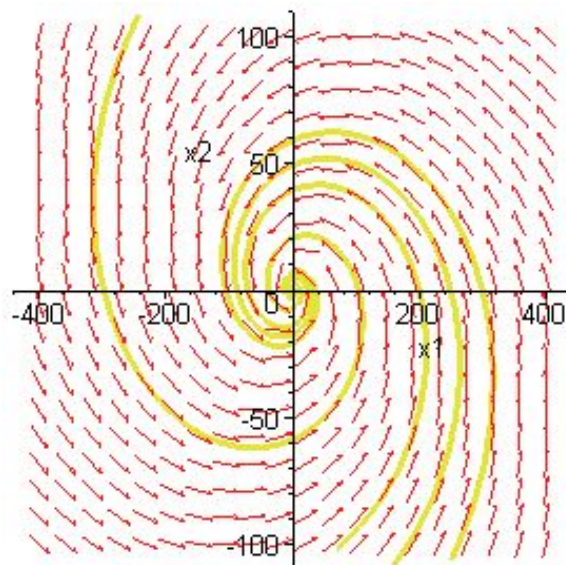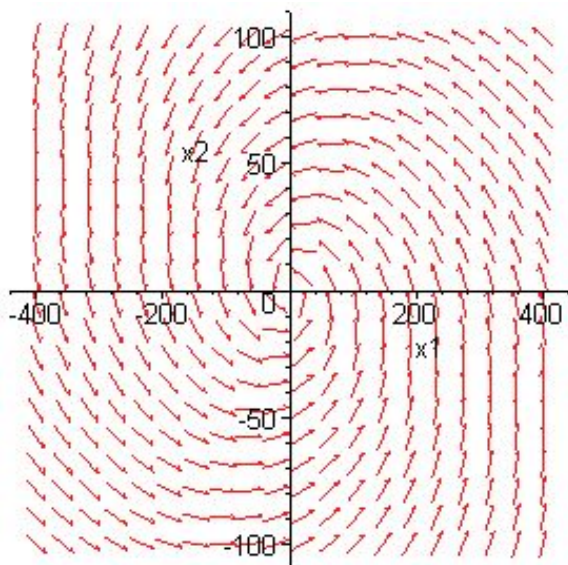

Plots of $x_1(t)$, $x_2(t)$ can be produced with maple code similar to this example:

```
DEplot({de},{x1(t),x2(t)},t=0..8,
       [[x1(0)=300,x2(0)=0]],scene=[t,x1],stepsize=0.05);
```


End of Maple Lab 10.