**Name** ______________________  Class Time _____

**Math 2250 Extra Credit Problems**
**Chapter 8**
**S2014**

**Submitted work**. Please submit one stapled package with this sheet on top. Kindly check-mark the problems submitted and label the paper **Extra Credit**. Label each solved problem with its corresponding problem number, e.g., Xc10.3-20.

## Problem Xc8.1-4. (Fundamental Matrix)

Find a fundamental matrix $\Phi(t)$ by each of the following methods. Report $e^{At} = \Phi(t)\Phi(0)^{-1}$, using one of the answers for $\Phi$.

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

**(a)** *Cayley-Hamilton method.* Compute the characteristic equation $\det(A - \lambda I) = 0$. Find two atoms from the roots of this equation. Then $x(t)$ is a linear combination of these atoms. The first equation $x' = 2x - 5y$ can be solved for $y$ to find the second answer.

**(b)** *Eigenanalysis method.* Find the eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$ of $A$. Let $\Phi$ have columns $e^{\lambda_1 t}\mathbf{v}_1$, $e^{\lambda_2 t}\mathbf{v}_2$.

## Problem Xc8.1-12. (Putzer's Method)

The exponential matrix $e^{At}$ can be found in the $2 \times 2$ case from Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the roots $\lambda_1$, $\lambda_2$ of $\det(A - \lambda I) = 0$ are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting $\lambda_2 \to \lambda_1$ [$\lambda_1$, $t$ fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute from Putzer's formula $e^{At}$ for the following cases.

**(a)** $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Answer $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

**(b)** $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

**(c)** $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

**(d)** $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

## Problem Xc8.1-38. (Laplace's Method)

The exponential matrix $e^{At}$ can be found from the Laplace resolvent formula for the problem $\Phi' = A\Phi$, $\Phi(0) = I$:

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1}\Phi(0) = (sI - A)^{-1}.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$, which implies $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Compute $\Phi(t) = e^{At}$ using the resolvent formula for the following cases.

**(a)** $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

**(b)** $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

**(c)** $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

### Problem Xc8.2-4. (Variation of Parameters)

Use the variation of parameters formula $\mathbf{u}_p(t) = e^{At}\int e^{-At}\mathbf{f}(t)dt$ to find a particular solution of the given system. Please use technology to do the indicated integration, following the example below.

**(a)** $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

**(b)** $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

**Example**: Solve for $\mathbf{u}_p(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

### Problem Xc8.2-19. (Initial Value Problem)

Solve the given initial value problem using a computer algebra system. Follow the example given below.

**(a)** $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

**(b)** $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

**Example**: Solve for $\mathbf{u}(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The answer is $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t):
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
```

**End of extra credit problems chapter 8.**