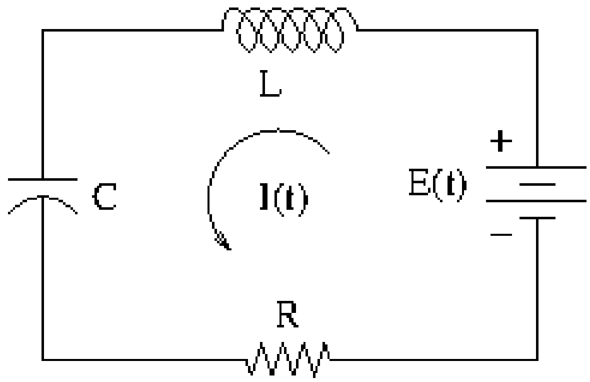
Sample Quiz 8

Sample Quiz 8, Problem 1. *RLC*-**Circuit**



The Problem. Suppose $E = 100\sin(20t)$, $L = 5$ H, $R = 250\,\Omega$ and $C = 0.002$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $5I'' + 250I' + 500I = 2000\cos(20t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 20$ is the input frequency, the natural frequency of $E = 100\sin(20t)$ and $E' = 2000\cos(20t)$.

(c) Substitute $I = A\cos(20t) + B\sin(20t)$ into the current model (a) and solve for $A = \frac{-12}{109}, B = \frac{40}{109}$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(20t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = \frac{4}{\sqrt{109}}$, $\delta = \arctan(3/10)$, $\delta/\omega = 0.01457$.

References

Course slides on Electric Circuits. Edwards-Penney *Differential Equations and Boundary Value Problems*, section 3.7, course supplement. EP or EPH sections 5.4, 5.5, 5.6.

Solutions to Problem 1

Problem 1(a) Start with $5Q'' + 250Q' + 500Q = 100\sin(20t)$. Differentiate across to get $5Q''' + 250Q'' + 500Q' = 2000\cos(20t)$. Change Q' to I.

Problem 1(b) $S = (20)(5) - 1/(20 * 0.002) = 75$

Problem 1(c) It helps to use the differential equation $u'' + 400u = 0$ satisfied by both $u_1 = \cos(20t)$ and $u_2 = \sin(20t)$. Functions u_1, u_2 are Euler solution atoms, hence independent. Along the solution path, we'll use $u_1' = -20\sin(20t) = -20u_2$ and $u_2' = 20\cos(20t) = 20u_1$. The arithmetic is simplified by dividing the equation first by 5. We then substitute $I = Au_1 + Bu_2$.

$$\begin{array}{l}
I'' + 50I' + 100I = 400\sin(20t) \\
A(u_1'' + 50u_1' + 100u_1) + B(u_2'' + 50u_2' + 100u_2) = 400\sin(20t) \\
A(-400u_1 + 50(-20u_2) + 100u_1) + B(-400u_2 + 50(20u_1) + 100u_2) = 400\sin(20t) \\
(-400A + 100A + 1000B)u_1 + (-1000A - 400B + 100B)u_2 = 400u_2
\end{array}$$

By independence of u_1, u_2, coefficients of u_1, u_2 on each side of the equation must match. The linear algebra property is called *unique representation of linear combinations*. This implies the 2×2 system of equations

$$\begin{array}{rcrcl}
-300A & + & 1000B & = & 0, \\
-1000A & - & 300B & = & 400.
\end{array}$$

The solution by Cramer's rule (the easiest method) is $A = -12/109, B = 40/109$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

The **steady-state current** is defined to be the sum of those terms in the general solution of the differential equation that remain after all terms that limit to zero at $t = \infty$ have been removed. The logic is that only these terms contribute to a graphic or to a numerical calculation after enough time has passed (as $t \to \infty$).

Problem 1(d) Let $\cos(\delta) = B/I_0, \sin(\delta) = -A/I_0, I_0 = \sqrt{A^2 + B^2}$. Use the trig identity

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

to rearrange the current formula as follows:

$$I(t) = A\cos(20t) + B\sin(20t) = I_0(\sin(20t)\cos(\delta) - \sin(\delta)\cos(20t)) = I_0\sin(20t - \delta).$$

Compute $I_0 = \sqrt{A^2 + B^2} = \frac{4}{\sqrt{109}}$. Compute $\tan(\delta) = \frac{\sin\delta}{\cos\delta} = -A/B = 12/40$. Then $\delta = \arctan(12/40)$ and finally $\delta/\omega = \arctan(3/10)/20 = 0.01457$.

Another method, using Edwards-Penney Differential Equations and Boundary Value Problems, section 3.7: Compute the **impedance** $Z = \sqrt{R^2 + S^2} = \sqrt{250^2 + 75^2} = \sqrt{68125} = 25\sqrt{109}$ and then $I_0 = E_0/Z = 4/\sqrt{109}$. The phase $\delta = \arctan(S/R) = \arctan(75/250) = \arctan(3/10)$. Then the time lag is $\delta/\omega = \frac{\arctan(0.3)}{20} = 0.01457$.

Sample Quiz 8, Problem 2. Picard's Theorem and RLC-Circuit Models

Picard-Lindelöf Theorem. Let $\vec{f}(x,\vec{y})$ be defined for $|x-x_0| \le h$, $\|\vec{y}-\vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &=& \vec{f}(x,\vec{y}(x)), \quad |x-x_0| < H, \\ \vec{y}(x_0) &=& \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x-x_0| < H$.

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The Problem. The second order problem

$$\text{(1)} \qquad \begin{cases} u'' + 2u' + 5u = 60\sin(20x), \\ u(0) = 1, \\ u'(0) = 0 \end{cases}$$

is an RLC-circuit charge model, in which the variables have been changed. The variables are time x in seconds and charge $u(x)$ in coulombs. Coefficients in the equation represent an inductor $L = 1$ H, a resistor $R = 2\,\Omega$, a capacitor $C = 0.2$ F and a voltage input $E(x) = 60\sin(20x)$.

The several parts below detail how to convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies. Please fill in the missing details.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = 0$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -5y_1 - 2y_2 + 60\sin(20x).$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (**??**) is

$$\text{(2)} \qquad \begin{cases} \vec{y}\,'(x) &=& \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}, \\ \vec{y}(0) &=& \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases}$$

(d) System (**??**) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x,\vec{y}) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

Solutions to Problem 2

(a) $\vec{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

(b) Differentiate, $y_1' = u'(x) = y_2$ and $y_2' = u''(x)$. Isolate u'' left in the equation $u''+2u'+5u = 60\sin(20x)$, then reduce $y_2' = u''(x)$ into $y_2' = -2u' - 5u + 60\sin(20x) = -2y_2 - 5y_1 + 60\sin(20x)$.

(c) Initial data $\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ was derived in part (a). The differential equation is derived from the scalar dynamical system in part (b), as follows.

$$\begin{aligned}
\vec{y}\,' &= \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 + 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}
\end{aligned}$$

(d) From calculus, polynomials and trigonometric function sine are infinitely differentiable. Therefore, in each of the variables x, y_1, y_2 the components of $\vec{f}$, which are just the right sides of the dynamical system equations of part (b), are also infinitely differentiable.

Sample Quiz 8, Problem 3. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y''+16y = 0$ has characteristic equation $r^2+16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
   The solution space S of the differential equation is given by

$$S = \mathbf{span}(\text{the } N \text{ Euler solution atoms}).$$

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$.[1]

Four complex conjugate roots listed according to multiplicity as $2\pm 3i, 2\pm 3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$, $xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a\pm bi$ $(b > 0)$ arising from roots of $(r-a)^2+b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 16y' = 0$

(b) $y'' + 16y = 0$

(c) $y'''' + 16y'' = 0$

(d) $y'' + 16y = 0$, $y(0) = 1$, $y'(0) = -1$

(e) $y'''' + 9y'' = 0$, $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r - 2)^2(r^2 - 4) = 0$.

(g) The characteristic equation is $(r - 1)^2(r^2 - 1)((r + 2)^2 + 4) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3 + 4i, 3 - 4i$.

[1] The Reason: $\cos(3x) = \frac{1}{2}e^{3xi} + \frac{1}{2}e^{-3xi}$ by Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Then $e^{2x}\cos(3x) = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$ is a linear combination of exponentials e^{rx} where r is a root of the characteristic equation. Euler's substitution implies e^{rx} is a solution, so by superposition, so also is $e^{2x}\cos(3x)$. Similar for $e^{2x}\sin(3x)$.

Solutions to Problem 3

(a) $y''+16y'=0$ upon substitution of $y=e^{rx}$ becomes $(r^2+16r)e^{rx}=0$. Cancel e^{rx} to find the **characteristic equation** $r^2+16r=0$. It factors into $r(r+16)=0$, then the two roots r make the list $r=0,-16$. The Euler solution atoms for these roots are e^{0x}, e^{-16x}. Report the general solution $y=c_1e^{0x}+c_2e^{-16x}=c_1+c_2e^{-16x}$, where symbols c_1, c_2 stand for arbitrary constants.

(b) $y''+16y=0$ has characteristic equation $r^2+16=0$. Because a quadratic equation $(r-a)^2+b^2=0$ has roots $r=a\pm bi$, then the root list for $r^2+16=0$ is $0+4i, 0-4i$, or briefly $\pm 4i$. The Euler solution atoms are $e^{0x}\cos(4x), e^{0x}\sin(4x)$. The general solution is $y=c_1\cos(4x)+c_2\sin(4x)$, because $e^{0x}=1$.

(c) $y''''+16y''=0$ has characteristic equation $r^4+4r^2=0$ which factors into $r^2(r^2+16)=0$ having root list $0,0,0\pm 4i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(4x), e^{0x}\sin(4x)$. Then the general solution is $y=c_1+c_2x+c_3\cos(4x)+c_4\sin(4x)$.

(d) $y''+16y=0$, $y(0)=1$, $y'(0)=-1$ defines a particular solution y. The usual arbitrary constants c_1, c_2 are determined by the initial conditions. From part (b), $y=c_1\cos(4x)+c_2\sin(4x)$. Then $y'=-4c_1\sin(4x)+4c_2\cos(4x)$. Initial conditions $y(0)=1, y'(0)=-1$ imply the equations $c_1\cos(0)+c_2\sin(0)=1, -4c_1\sin(0)+4c_2\cos(0)=-1$. Using $\cos(0)=1$ and $\sin(0)=0$ simplifies the equations to $c_1=1$ and $4c_2=-1$. Then the particular solution is $y=c_1\cos(4x)+c_2\sin(4x)=\cos(4x)-\frac{1}{4}\sin(4x)$.

(e) $y''''+9y''=0$, $y(0)=y'(0)=0$, $y''(0)=y'''(0)=1$ is solved like part (d). First, the characteristic equation $r^4+9r^2=0$ is factored into $r^2(r^2+9)=0$ to find the root list $0,0,0\pm 3i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(3x), e^{0x}\sin(3x)$, which implies the general solution $y=c_1+c_2x+c_3\cos(3x)+c_4\sin(3x)$. We have to find the derivatives of y: $y'=c_2-3c_3\sin(3x)+3c_4\cos(3x)$, $y''=-9c_3\cos(3x)-9c_4\sin(3x)$, $y'''=27c_3\sin(3x)-27c_4\cos(3x)$. The initial conditions give four equations in four unknowns c_1, c_2, c_3, c_4:

$$\begin{array}{ccccccccc}
c_1 & + & c_2(0) & + & c_3\cos(0) & + & c_4\sin(0) & = & 0,\\
 & & c_2 & - & 3c_3\sin(0) & + & 3c_4\cos(0) & = & 0,\\
 & & & - & 9c_3\cos(0) & - & 9c_4\sin(0) & = & 1,\\
 & & & & 27c_3\sin(0) & - & 27c_4\cos(0) & = & 1,
\end{array}$$

which has invertible coefficient matrix $\begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 3\\ 0 & 0 & -9 & 0\\ 0 & 0 & 0 & -27\end{pmatrix}$ and right side vector $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$. The solution is $c_1=c_2=1/9$, $c_3=-1/9, c_4=-1/27$. Then the particular solution is $y=c_1+c_2x+c_3\cos(3x)+c_4\sin(3x)=\frac{1}{9}+\frac{1}{9}x-\frac{1}{9}\cos(3x)-\frac{1}{27}\sin(3x)$

(f) The characteristic equation is $(r-2)^2(r^2-4)=0$. Then $(r-2)^3(r+2)=0$ with root list $2,2,2,-2$ and Euler atoms $e^{2x}, xe^{2x}, x^2e^{2x}, e^{-2x}$. The general solution is a linear combination of these four atoms.

(g) The characteristic equation is $(r-1)^2(r^2-1)((r+2)^2+4)=0$. The root list is $1,1,1,-1,-2\pm 2i$ with Euler atoms $e^x, xe^x, x^2e^x, e^{-x}, e^{-2x}\cos(2x), e^{-2x}\sin(2x)$. The general solution is a linear combination of these six atoms.

(h) The characteristic equation roots, listed according to multiplicity, are $0,0,0,-1,2,2,3+4i,3-4i$. Then the Euler solution atoms are $e^{0x}, xe^{0x}, x^2e^{0x}, e^{-x}, e^{2x}, xe^{2x}, e^{3x}\cos(4x), e^{3x}\sin(4x)$. The general solution is a linear combination of these eight atoms.