Sample Quiz 5

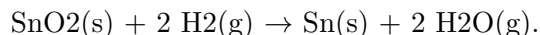**Sample Quiz5 Problem 1.** **Balancing Chemical Equations**: Tin oxide heated with hydrogen gas forms tin metal and water vapor.

The object of this problem is the derivation of the balanced chemical equation
$$SnO2(s) + 2\ H2(g) \rightarrow Sn(s) + 2\ H2O(g).$$

The chemical symbols are defined as follows:

|			
SnO2(s)	Tin dioxide	(s)=solid state	Ref: Tin dioxide
H2(g)	Hydrogen	(g)=gaseous state	Ref: Hydrogen
Sn(s)	Tin	(s)=solid state	Ref: Tin
H2O(g)	Water	(g)=gaseous state	Ref: Water vapor

Chemical balance means that matching elements on each side of a chemical equation have the same total amount of atoms. Notation 2 H2O documents 2x2=4 hydrogen atoms and 2x1=2 oxygen atoms. The given chemical equation is balanced, because the atom amounts for elements Sn, H, O total on each side of the equation to 1, 4, 2, respectively.

Chemical Balance by Linear Algebra. We begin with the unbalanced chemical equation
$$SnO2(s) + H2(g) \rightarrow Sn(s) + H2O(g),$$
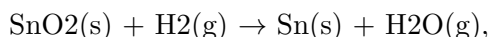
which means tin dioxide and hydrogen gas chemically combine into tin and water vapor.

Let's write the balanced equation as
$$x_1\ SnO2(s) + x_2\ H2(g) \rightarrow x_3\ Sn(s) + x_4\ H2O(g),$$
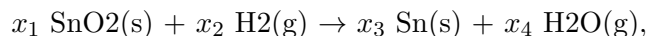
where integers x_1 to x_4 are to be determined. Chemical balance means that the following algebraic equations hold:

$$
\begin{aligned}
x_1 &= x_3 & \text{Tin Sn} \\
2x_2 &= 2x_4 & \text{Hydrogen H} \\
2x_1 &= x_4 & \text{Oxygen O}
\end{aligned}
$$

(a) Write the equations in standard matrix form $A\vec{x} = \vec{0}$, where A is a 3×4 matrix and $\vec{x}$ has components x_1 to x_4.

(b) Show details for computing the reduced row echelon form $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$

(c) There are more variables than equations. Therefore, there is at least one free variable. Identify the lead variables and the free variables.

(d) Solve the system $A\vec{x} = \vec{0}$. The answer is $x_1 = \frac{1}{2}t_1$, $x_2 = t_1$, $x_3 = \frac{1}{2}t_1$, $x_4 = t_1$, in terms of invented symbol t_1.

Balanced Equation. The smallest positive integer values for x_1 to x_4 are obtained from $t_1 = 2$. Then $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$. Substitution gives the balanced chemical equation
$$(1)SnO2(s) + (2)H2(g) \rightarrow (1)Sn(s) + (2)H2O(g)$$
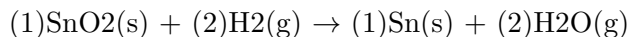
References. Edwards-Penney sections 3.1, 3.2, 3.3. Especially, re-read 3.3 Theorem 3: *If variable $\vec{x}$ has more entries than there are equations in system $A\vec{x} = \vec{0}$, then the system has infinitely many solutions.* Course document Linear Algebraic Equations, No Matrices. How to balance chemical equations, a video by Anne Marie Helmenstine with English transcript.

Sample Quiz5 Problem 2. Solving Higher Order Initial Value Problems with Linear Algebra.

The objective of this problem is to learn how to solve for the constants in a general solution, using given initial conditions. The differential equation and its known general solution are

$$y''' + 4y'' = 0,$$
$$y(x) = c_1 + c_2 x + c_3 e^{-4x}.$$

The initial conditions are

$$y(0) = 1, y'(0) = 2, y''(0) = -1.$$

(a) Derive the general solution formula, as follows. Make the substitution $v = y''$ to obtain $v' + 4v = 0$. Solve for v and then integrate twice to find y.

(b) Substitute the general solution into the three initial conditions to obtain the system of equations

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(c) Solve the system of equations and then report the particular solution $y(x)$ so found.

> **Remark.** Picard's existence-uniqueness theorem says that the initial value problem has a unique solution, meaning that the linear algebra problem is in the unique solution case: no free variable.

References. Edwards-Penney sections 3.3, 3.4, 5.3. Course manuscript Linear Algebraic Equations, No Matrices.

Sample Quiz5 Problem 3. An RL-Circuit with DC Voltage Source.
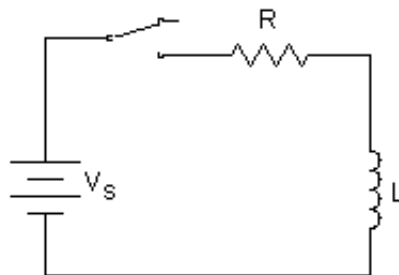
The RL-circuit in the figure has model $LI' + RI = V_s$ where symbols L, R, V_s are constants. If the charge is $Q(t)$, then $I = Q'$ and $LQ'' + RQ' = V_s$. Differentiate this equation to obtain $LQ''' + RQ'' = 0$. Assume the current and charge are initially zero, before the switch is closed.

(a) Explain why Q has initial conditions $Q(0) = Q'(0) = 0$, $Q''(0) = V_s/L$.

(b) Solve for the charge $Q(t)$ and current $I(t)$ from the higher order equation $LQ''' + RQ'' = 0$ with initial conditions $Q(0) = Q'(0) = 0$, $Q''(0) = V_s/L$, guided by Problem 2, above. The system of linear equations to be solved is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{R}{L} \\ 0 & 0 & \frac{R^2}{L^2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{V_s}{L} \end{pmatrix}.$$

(c) The steady-state current $\lim_{t \to \infty} I(t)$ computed directly from your solution formula should equal the answer from the equilibrium solution of $LI' + RI = V_s$. Does it?

References. Edwards-Penney sections 3.3, 3.4, 5.3. Course manuscript Linear Algebraic Equations, No Matrices. Edwards-Penney BVP 3.7, electric circuit supplement. All About Circuits Volume I – DC by T. Kuphaldt. Course slides on Electric Circuits.