Sample Quiz 12

Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a one-day trip from Salt Lake City to Pine Bluffs, Wyoming, on the Nebraska border. The route is I-80 E, 471 miles through Utah and Wyoming. Google maps estimates 6 hours and 52 minutes driving time. The table below shows the distances, time, road segment and average speed with total trip time 7 hours and 42 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.5	Kimball to Wanship	$65 - 55$
9.1	13	42	Wanship to Coalville	70
5.7	7	48.9	Coalville to Echo Dam	70
16.5	18	55	Echo Dam to 75 mph sign	70
39	36	65	75 mph sign to Evanston	75
308	269	68.7	Evanston to Laramie	75
50.6	50	61.2	Laramie to Cheyenne	75
43	37	69.7	Cheyenne to Pine bluffs	75

The **velocity function** for the SUV is approximated by

$$V_{\mathrm{pc}}(t) = \begin{cases} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\[4pt] 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.5 & 20 < t < 32 & \text{Kimball to Wanship} \\ 42.0 & 32 < t < 45 & \text{Wanship to Coalville} \\ 48.9 & 45 < t < 52 & \text{Coalville to Echo Dam} \\ 55.0 & 52 < t < 70 & \text{Echo Dam to 75 mph sign} \\ 65.0 & 70 < t < 106 & \text{75 mph sign to Evanston} \\ 68.7 & 106 < t < 375 & \text{Evanston to Laramie} \\ 61.2 & 375 < t < 425 & \text{Laramie to Cheyenne} \\ 69.7 & 425 < t < 462 & \text{Cheyenne to Pine bluffs} \\ 0.00 & 462 < t < \infty & \text{SUV stopped} \end{cases}$$

The velocity function $V_{\mathrm{pc}}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \begin{cases} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{cases}$$

where functions $f_1, f_2, \ldots, f_n$ are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(t_k) = \lim_{h \to 0+} f(t_k + h)$, which makes the function right-continuous.

The Problem. The SUV travels from $t = 0$ to $t = \frac{462}{60} = 7.7$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{\mathrm{pc}}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{\mathrm{pc}}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\mathrm{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Solution.

Method of Quadrature. The meaning of the differential equation is that $x'(t)$ is piecewise continuous. We want $x(t)$ to be continuous, because it is the odometer trip meter reading. But $x'(t)$ cannot be continuous, if we require $\frac{dx}{dt} = V_{\text{pc}}(60t)$, because the right side is piecewise defined and discontinuous at division points.

Theorem (Fundamental Theorem of Calculus)
If $f'(x)$ is piecewise continuous and $f(x)$ is continuous on $a \le x \le b$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

The theorem implies that the method of quadrature works for the equation $x'(t) = V_{\text{pc}}(60t)$. The quadrature method gives the correct answer

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

Another plan is to split $x'(t) = V_{\text{pc}}(60t)$ into 10 simple equations, $x' = 54.3$, $x(0) = 0$ on $0 \le t < 20$ being the first equation. The next equation is $x' = 56.5$, $x(20) = x_0$, on $20 < t < 32$. To make $x(t)$ continuous, we must choose $x_0 = 1086$, which is the value at the division point $t = 20$ assumed by the first problem ($x' = 54.3$, $x(0) = 0$ on $0 \le t < 20$). This tedious process has to be continued for all 10 segments. The result is that $x(t)$ is **piecewise linear** between division points.

Laplace's Method. The piecewise continuous input $V_{\text{pc}}(60t)$ is of exponential order, because it is zero after $t = 462/60$. Laplace theory says it has a Laplace transform $\mathcal{L}(V_{\text{pc}}(60t))$. Assuming a continuous solution $x(t)$, with $x'(t)$ piecewise continuous, then the equation to be satisfied is

$$s\,\mathcal{L}(x(t)) - x(0) = \mathcal{L}(x'(t)) = \mathcal{L}(V_{\text{pc}}(60t)).$$

The **Laplace integral theorem** implies

$$\mathcal{L}(x(t)) = \frac{1}{s}\mathcal{L}(V_{\text{pc}}(60t)) = \mathcal{L}\left(\int_0^t V_{\text{pc}}(60w)dw\right).$$

Lerch's theorem then implies that the symbol $\mathcal{L}$ cancel from each side, giving the odometer trip meter reading in terms of the integral of the piecewise continuous input $V_{\text{pc}}(60t)$:

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

We'll use technology to program and evaluate the integral, even though it can be done by hand. The plan is to plot the trip meter reading, then comment on the slow and fast segments of the route, by using a clever plot involving the average speed. The last display is the piecewise linear trip meter reading $x(t)$.

Maple

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.5, t < 45, 42,  t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
plot(X(t),t=0..480/60); # Almost a straight line.
```

Average Speed

Define the average value of a function $f(w)$ on $a \le w \le b$ by $\frac{1}{b-a}\int_a^b f(w)dw$. Then the average speed in the example is

$$\frac{\int_0^{462/60} V_{\text{pc}}(60w)dw}{462/60} = 65.14956710.$$

A Clever Plot

An average driver would try to maintain 65.15 mph. The clever plot will create a graphic of $x(t)-65.15t$ on interval $0 \le t \le T_1$, where T_1 is the 471 mile trip time at 65.15 mph.

```
# Maple code
 Xpc:=t->piecewise(t<0,0,
 t < 20 ,54.3, t < 32, 56.5, t < 45, 42,  t < 52, 48.9,
 t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
 t < 462, 69.7, 0.0);
 X:=t->int(Xpc(60*w),w=0..t);
 AVEspeed:=X(462/60)/(462/60); # AVEspeed = 65.14956710 mph
 T1:=solve(AVEspeed*t=471,t);   # T1 = 7.229518491 hours
 plot(X(t)-AVEspeed*t,t=0..T1);
```
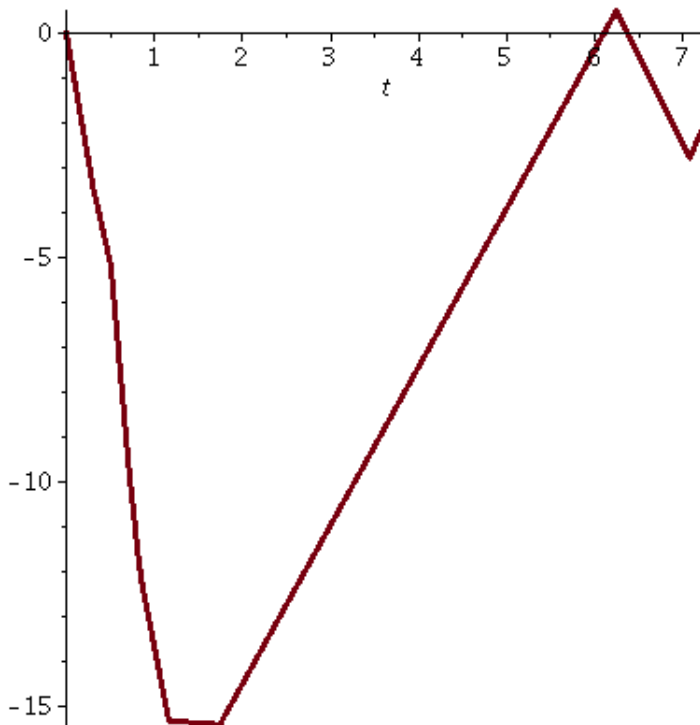

We see from the graphic that segments of the road cause a slowdown of up to 15 mph, but for a brief interval it is possible to exceed the average speed, due to a 75 mph speed limit.

```
# Maple code for piecewise linear display
X:=t->int(Xpc(60*w),w=0..t);
convert(X(t),piecewise,t):evalf(%,4);
```

$$
\text{Trip meter at time } t = \begin{cases}
0.0 & t \le 0.0 \\
54.30\,t & t \le 0.3333 \\
56.30\,t - 0.6667 & t \le 0.53 \\
42.0\,t + 6.960 & t \le 0.75 \\
48.90\,t + 1.785 & t \le 0.8667 \\
55.0\,t - 3.502 & t \le 1.167 \\
65.0\,t - 15.17 & t \le 1.767 \\
68.70\,t - 21.70 & t \le 6.25 \\
61.20\,t + 25.17 & t \le 7.083 \\
69.70\,t - 35.04 & t \le 7.7 \\
501.7 & 7.7 < t
\end{cases}
$$

Problem 2. Switches and Impulses

Laplace's method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step   Define $u(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp   Define **ramp**$(t - a) = (t - a)u(t - a) = \begin{cases} t - a & t \geq a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse   Define **pulse**$(t, a, b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t - a) - u(t - b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \leq t \leq b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b - a} \right)(b - a) = \text{Average Force} \times \text{Duration Time.}$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t - a) = \dfrac{du}{dt}(t - a)$, where $u(t - a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t - a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t - a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\,\textbf{pulse}(t, a - h, a + h).$$

- The fundamental equation is $\int_{-\infty}^{\infty} F(x)\delta(x - a)dx = F(a)$. Symbol $\delta(t - a)$ is not manipulated as an ordinary function, but regarded as $du(t - a)/dt$ in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \geq 0$,

Forward table	Backward table
$\mathcal{L}\left(f(t - a)u(t - a)\right) = e^{-as}\,\mathcal{L}(f(t))$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t - a)u(t - a)\right)$
$\mathcal{L}(g(t)u(t - a)) = e^{-as}\,\mathcal{L}\left(g(t)\big|_{t = t + a}\right)$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)\big|_{t = t - a}\right).$

The Problem. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

$$\text{(1)} \ \mathcal{L}(10u(t - \pi)) \quad \text{(2)} \ \mathcal{L}(\mathbf{ramp}(t - 2\pi)), \quad \text{(3)} \ \mathcal{L}(10\,\mathbf{pulse}(t, 3, 5)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$\text{(1)} \ \mathcal{L}(f) = \frac{5e^{-3s}}{s} \quad \text{(2)} \ \mathcal{L}(f) = \frac{e^{-4s}}{s^2} \quad \text{(3)} \ \mathcal{L}(f) = \frac{5}{s}\left(e^{-2s} - e^{-3s}\right).$$

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

$$\text{(1)} \ \mathcal{L}(10\delta(t - \pi)), \quad \text{(2)} \ \mathcal{L}(5\delta(t-1) + 10\delta(t-2) + 15\delta(t-3)), \quad \text{(3)} \ \mathcal{L}((t-\pi)\delta(t-\pi)).$$

The sum of Dirac impulses in (2) is called an **impulse train**.

Solutions

Solution (a). The forward second shifting theorem applies.

(1) $\mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a))$ where $g(t) = 10$ and $a = \pi$. Then $\mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) = e^{-as}\mathcal{L}\left(g(t)|_{t=t+a}\right) = e^{-\pi s}\mathcal{L}\left(10|_{t=t+\pi}\right) = \frac{10}{s}e^{-\pi s}$.

(2) $\mathcal{L}(\mathbf{ramp}(t - 2\pi)) = \mathcal{L}((t - 2\pi)u(t - 2\pi)) = \mathcal{L}\left(tu(t)|_{t=t-2\pi}\right) = e^{-2\pi s}\mathcal{L}(t) = \frac{1}{s^2}e^{-2\pi s}$.

(3) $\mathcal{L}(10\,\mathbf{pulse}(t, 3, 5)) = 10\,\mathcal{L}(u(t - 3) - u(t - 5)) = \frac{10}{s}\left(e^{3s} - e^{-5s}\right)$.

Solution (b). Presence of an exponential e^{-as} signals unit step $u(t - a)$ in the answer, the main tool being the backward second shifting theorem.

(1) $\mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s}\frac{5}{s} = e^{-3s}\mathcal{L}(5) = \mathcal{L}(5u(t)|_{t=t+3}) = \mathcal{L}(5u(t - 3))$. Lerch implies $f = 5u(t - 3)$.

(2) $\mathcal{L}(f) = \frac{e^{-4s}}{s^2} = \frac{e^{-as}}{\mathcal{L}}(t)$ where $a = 4$. Then $\mathcal{L}(f) = \frac{e^{-as}}{\mathcal{L}}(t) = \mathcal{L}(tu(t)|_{t=t-a}) = \mathcal{L}((t - 4)u(t - 4)) = \mathcal{L}(\mathbf{ramp}(t - 4))$. Lerch implies $f = \mathbf{ramp}(t - 4)$.

(3) $\mathcal{L}(f) = e^{-2s}\frac{5}{s} - e^{-3s}\frac{5}{s} = \mathcal{L}(5u(t - 2)) - \mathcal{L}(5u(t - 3)) = \mathcal{L}(5\,\mathbf{pulse}(t, 2, 3))$. Lerch implies $f = 5\,\mathbf{pulse}(t, 2, 3)$.

Solution (c). The main result for Dirac unit impulse δ is the equation

$$\int_{0}^{\infty} g(t)\delta(t - a)dt = g(a),$$

valid for $g(t)$ continuous on $0 \leq t < \infty$. When $g(t) = e^{-st}$, then the equation implies the Laplace formula $\mathcal{L}(\delta(t - a)) = e^{-as}$.

(1) $\mathcal{L}(10\delta(t - \pi)) = 10e^{-\pi s}$, by the displayed equation with $g(t) = 10e^{-st}$, or by using linearity and the formula $\mathcal{L}(\delta(t - a)) = e^{-as}$.

(2) $\mathcal{L}(5\delta(t-1) + 10\delta(t-2) + 15\delta(t-3)) = 5\,\mathcal{L}(\delta(t-1)) + 10\,\mathcal{L}(\delta(t-2)) + 15\,\mathcal{L}(\delta(t-3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s}$.

(3) $\mathcal{L}((t-\pi)\delta(t-2\pi)) = \int_0^\infty (t-\pi)e^{st}\delta(t-2\pi)dt = (t-\pi)e^{-st}|_{t=2\pi} = \pi e^{-2\pi s}$, using $g(t) = (t-\pi)e^{-st}$ and $a = 2\pi$ in the equation.

The Riemann-Stieltjes Integral

Definition

The Riemann-Stieltjes integral of a real-valued function f of a real variable with respect to a real monotone non-decreasing function g is denoted by

$$\int_a^b f(x)\,dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating RiemannStieltjes sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the **integrand** and the **integrator**.

The **limit** is a number A, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\mathbf{mesh}(P) = \max_{0 \le i < n} (x_{i+1} - x_i) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$|S(P, f, g) - A| < \varepsilon.$$

Problem 3. Experiment to Find the Transfer Function $h(t)$

Consider a second order problem

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

which by Laplace theory has a particular solution solution defined as the convolution of the transfer function $h(t)$ with the input $f(t)$,

$$x_p(t) = \int_0^t f(w)h(t-w)dw.$$

Examined in this problem is another way to find $h(t)$, which is the system response to a Dirac unit impulse with zero data. Then $h(t)$ is the solution of

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

The Problem. Assume a, b, c are constants and define $g(t) = \int_0^t h(w)dw$.

(a) Show that $g(0) = g'(0) = 0$, which means g has zero data.

(b) Let $u(t)$ be the unit step. Argue that g is the solution of

$$ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.$$

The fundamental theorem of calculus says that $h(t) = g'(t)$. Therefore, to compute the transfer function $h(t)$, find the response $g(t)$ to the unit step with zero data, followed by computing the derivative $g'(t)$, which equals $h(t)$.

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function $h(t)$ in the special case

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

Solutions

(a) $g(0) = \int_0^0 h(w)dw = 0$, $g'(0) = h'(0) = 0$.

(b) Let $u(t)$ be the unit step. Initial data was decided in part (a). The Laplace applied to $ag''(t) + bg'(t) + cg(t) = u(t)$ gives $(as^2 + bs + c)\mathcal{L}(g) = \mathcal{L}(u(t))$. Then $\mathcal{L}(g) = \mathcal{L}(h(t))\mathcal{L}(u(t)) = \mathcal{L}(h(t))\frac{1}{s}\mathcal{L}\left(\int_0^t h(r)du\right)$ by the **integral theorem**. Lerch's theorem then says $g(t) = \int_0^t h(r)dr$.

(c) For equation $x''(t) + 2x'(t) + 5x(t) = f(t)$ we replace $x(t)$ by $g(t)$ and $f(t)$ by the unit step $u(t)$, then solve $g''(t) + 2g'(t) + 5g(t) = u(t)$, obtaining $\mathcal{L}(g) = \frac{1}{s}\frac{1}{s^2+2s+5} = \mathcal{L}(\frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t)))$. Then $g(t) = \frac{1}{5} - \frac{1}{10}e^{-t}(2*\cos(2t) + \sin(2t))$ and $h(t) = g'(t) = \frac{1}{2}e^{-t}\sin(2t)$.