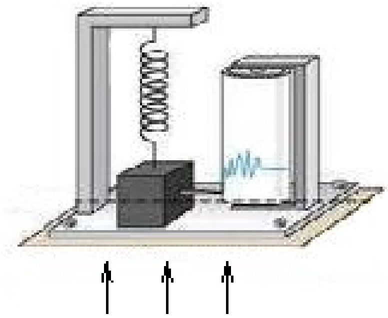
Sample Quiz 11

Sample Quiz 11, Problem 1. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$(1) \quad \begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium}, \\ u(t) = \text{vertical ground motion due to the earthquake}. \end{cases}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

(a) Explain how a **release test** can find values for β, Ω_0 in the model $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$.

(b) Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = Ce^{-at}\sin(bt)$ for some positive constants C, a, b. Assume a release test determined $2\beta\Omega_0 = 12$ and $\Omega_0^2 = 100$. Explain how to find a formula for the ground motion $u(t)$, then provide a formula for $u(t)$, using technology.

Solution.

(a) A **release test** is an experiment which provides initial data $x(0) > 0$, $x'(0) = 0$ to the seismoscope mass. The model is $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ (ground motion zero). The recorder graphs $x(t)$ during the experiment, until two successive maxima $(t_1, x_1), (t_2, x_2)$ appear in the graph. This is enough information to find values for β, Ω_0.

In an under-damped oscillation, the characteristic equation is $(r+p)^2 + \omega^2 = 0$ corresponding to complex conjugate roots $-p \pm \omega i$. The phase-amplitude form is $x(t) = Ce^{-pt}\cos(\omega t - \alpha)$, with period $2\pi/\omega$.

The equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ has characteristic equation $(r+\beta)^2 + \Omega_0^2 = 0$. Therefore $x(t) = Ce^{-\beta t}\cos(\Omega_0 t - \alpha)$.

The period is $t_2 - t_1 = 2\pi/\Omega_0$. Therefore, Ω_0 is known. The maxima occur when the cosine factor is 1, therefore

$$\frac{x_2}{x_1} = \frac{Ce^{-\beta t_2}\cdot 1}{Ce^{-\beta t_1}\cdot 1} = e^{-\beta(t_2 - t_1)}.$$

This equation determines β.

(b) The equation $-u''(t) = x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t)$ (the model written backwards) determines $u(t)$ in terms of $x(t)$. If $x(t)$ is known, then this is a quadrature equation for the ground motion $u(t)$.

For the example $x(t) = Ce^{-at}\sin(bt)$, $2\beta\Omega_0 = 12, \Omega_0^2 = 100$, then the quadrature equation is

$$-u''(t) = x''(t) + 12x'(t) + 100x(t).$$

After substitution of $x(t)$, the equation becomes

$$-u''(t) = C\mathrm{e}^{-at}\left(\sin(bt)\,a^2 - \sin(bt)\,b^2 - 2\cos(bt)\,ab - 12\sin(bt)\,a + 12\cos(bt)\,b + 100\sin(bt)\right)$$

which can be integrated twice using Maple, for simplicity. All integration constants will be assumed zero. The answer:

$$\begin{aligned} u(t) &= \frac{C\mathrm{e}^{-at}\left(12\,a^2 b + 12\,b^3 - 200\,ab\right)\cos(bt)}{\left(a^2+b^2\right)^2} \\ &\quad - \frac{C\mathrm{e}^{-at}\left(a^4 + 2\,a^2 b^2 + b^4 - 12\,a^3 - 12\,ab^2 + 100\,a^2 - 100\,b^2\right)\sin(bt)}{\left(a^2+b^2\right)^2} \end{aligned}$$

The Maple session has this brief input:

```
de:=-diff(u(t),t,t) = diff(x(t),t,t) + 12*diff(x(t),t) + 100* x(t);
x:=t->C*exp(-a*t)*sin(b*t);
dsolve(de,u(t));subs(_C1=0,_C2=0,%);
```

Sample Quiz 11, Problem 2. **Laplace Theory**

Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace's method **solves differential equations**.

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)$.

(b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16}.$$

(c) Solve the initial value problem $x''(t) + 256x(t) = 1$, $x(0) = 1$, $x'(0) = 0$.

Solution (a).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \mathcal{L}(te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)) & \\
 & = & \mathcal{L}(te^{2t}) + 2\,\mathcal{L}(t\sin(3t)) + 3\,\mathcal{L}(e^{-t}\cos(4t)) & \text{Linearity} \\
 & = & -\frac{d}{ds}\,\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\,\mathcal{L}(\sin(3t)) + 3\,\mathcal{L}(e^{-t}\cos(4t)) & \text{Differentiation rule} \\
 & = & -\frac{d}{ds}\,\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\,\mathcal{L}(\sin(3t)) + 3\,\mathcal{L}(\cos(4t))|_{s=s+1} & \text{Shift rule} \\
 & = & -\frac{d}{ds}\frac{1}{s-2} - 2\frac{d}{ds}\frac{3}{s^2+9} + 3\left.\frac{s}{s^2+16}\right|_{s=s+1} & \text{Forward table} \\
 & = & \frac{1}{(s-2)^2} + \frac{12s}{(s^2+9)^2} + 3\frac{s+1}{(s+1)^2+16} & \text{Calculus}
\end{array}$$

Solution (b).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16} & \\
 & = & 8\frac{2}{s^2+4} + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\frac{4}{s^2+16} & \text{Prep for backward table} \\
 & = & 8\,\mathcal{L}(\sin 2t) + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\,\mathcal{L}(\sin 4t) & \text{backward table} \\
 & = & 8\,\mathcal{L}(\sin 2t) + \left.\frac{s+2}{s^2+9}\right|\, s = s-1 + \frac{1}{2}\,\mathcal{L}(\sin 4t) & \text{shift rule} \\
 & = & 8\,\mathcal{L}(\sin 2t) + \mathcal{L}(\cos 3t + \frac{2}{3}\sin 3t)\Big|\, s = s-1 + \frac{1}{2}\,\mathcal{L}(\sin 4t) & \text{backward table} \\
 & = & 8\,\mathcal{L}(\sin 2t) + \mathcal{L}(e^t\cos 3t + e^t\frac{2}{3}\sin 3t) + \frac{1}{2}\,\mathcal{L}(\sin 4t) & \text{shift rule} \\
 & = & \mathcal{L}(8\sin 2t) + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t) & \text{Linearity} \\
f(t) & = & 8\sin 2t + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t & \text{Lerch's cancel rule}
\end{array}$$

Solution (c).

$$\begin{array}{lcll}
\mathcal{L}(x''(t) + 256x(t)) & = & \mathcal{L}(1) & \mathcal{L}\text{ acts like matrix mult} \\
s\,\mathcal{L}(x') - x'(0) + 256\,\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s(s\,\mathcal{L}(x) - x(0)) - x'(0) + 256\,\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s^2\,\mathcal{L}(x) - s + 256\,\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Use } x(0)=1, x'(0)=0 \\
(s^2+256)\,\mathcal{L}(x) & = & s + \mathcal{L}(1) & \text{Collect } \mathcal{L}(x) \text{ left}
\end{array}$$

$$\begin{array}{rcll}
\mathcal{L}(x) & = & \frac{s+\mathcal{L}(1)}{(s^2+256)} & \text{Isolate } \mathcal{L}(x) \text{ left} \\
\mathcal{L}(x) & = & \frac{s+1/s}{(s^2+256)} & \text{Forward table} \\
\mathcal{L}(x) & = & \frac{s^2+1}{s(s^2+256)} & \text{Algebra} \\
\mathcal{L}(x) & = & \frac{A}{s} + \frac{Bs+C}{s^2+256} & \text{Partial fractions} \\
\mathcal{L}(x) & = & A\,\mathcal{L}(1) + B\,\mathcal{L}(\cos 16t) + \frac{C}{16}\,\mathcal{L}(\sin 16t) & \text{Backward table} \\
\mathcal{L}(x) & = & \mathcal{L}(A + B\cos 16t + \frac{C}{16}\sin 16t) & \text{Linearity} \\
x(t) & = & A + B\cos 16t + \frac{C}{16}\sin 16t & \text{Lerch's rule}
\end{array}$$

The partial fraction problem remains:

$$\frac{s^2+1}{s(s^2+256)} = \frac{A}{s} + \frac{Bs+C}{s^2+256}$$

This problem is solved by clearing the fractions, then swapping sides of the equation, to obtain

$$A(s^2+256) + (Bs+C)(s) = s^2+1.$$

Substitute three values for s to find 3 equations in 3 unknowns A, B, C:

$$\begin{array}{lllr} s=0 & 256A & = & 1 \\ s=1 & 257A+B+C & = & 2 \\ s=-1 & 257A+B-C & = & 2 \end{array}$$

Then $A = 1/256, B = 255/256, C = 0$ and finally

$$x(t) = A + B\cos 16t + \frac{C}{16}\sin 16t = \frac{1+255\cos 16t}{256}$$

Answer Checks

```
# Sample quiz 11
# answer check problem 2(a)
f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
with(inttrans): # load laplace package
laplace(f,t,s);
# The last two fractions simplify to 3(s+1)/((s+1)^2+16).
# answer check problem 2(b)
F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
invlaplace(F,s,t);
# answer check problem 2(c)
de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
dsolve([de,ic],x(t));
# answer check problem 2(c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

The output appears on the next page

```
> # Sample quiz 11
> # answer check problem 2(a)
> f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
```

$$f := t\,e^{2t} + 2\,t\sin(3\,t) + 3\,e^{-t}\cos(4\,t) \tag{1}$$

```
> with(inttrans): # load laplace package
> laplace(f,t,s) assuming s::real;
```

$$\frac{1}{(s-2)^2} + \frac{12\,s}{(s^2+9)^2} + \frac{3}{2\,(s+1-4\,I)} + \frac{3}{2\,(s+1+4\,I)} \tag{2}$$

```
> # The last two fractions simplify to 3(s+1)/((s+1)^2+16).
> # answer check problem 2(b)
> F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
```

$$F := \frac{16}{s^2+4} + \frac{s+1}{s^2-2\,s+10} + \frac{2}{s^2+16} \tag{3}$$

```
> invlaplace(F,s,t);
```

$$8\sin(2\,t) + \frac{1}{2}\sin(4\,t) + \frac{1}{3}\,e^{t}\,(3\cos(3\,t) + 2\sin(3\,t)) \tag{4}$$

```
> # answer check problem 2(c)
> de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
```

$$de := \frac{d^2}{dt^2}\,x(t) + 256\,x(t) = 1$$

$$ic := x(0) = 1,\ D(x)(0) = 0 \tag{5}$$

```
> dsolve([de,ic],x(t));
```

$$x(t) = \frac{1}{256} + \frac{255}{256}\cos(16\,t) \tag{6}$$

```
> # answer check problem 2(c), partial fractions
> convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

$$\frac{1}{256\,s} + \frac{255}{256}\,\frac{s}{s^2+256} \tag{7}$$