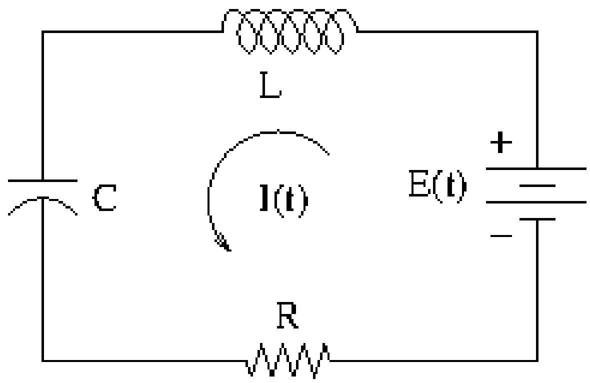
Quiz 8

Please attempt all parts of the three problems. You will receive full credit for a problem, if it is 60 percent completed.

Quiz 8, Problem 1. *RLC*-**Circuit**



The Problem. Suppose $E = \sin(40t)$, $L = 1$ H, $R = 50\,\Omega$ and $C = 0.01$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $I'' + 50I' + 100I = 40\cos(40t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 40$ is the input frequency, the natural frequency of $E = \sin(40t)$ and $E' = 40\cos(40t)$. Then find the **impedance** $Z = \sqrt{S^2 + R^2}$.

(c) The steady-state current is $I(t) = A\cos(40t) + B\sin(40t)$ for some constants A, B. Substitute $I = A\cos(40t) + B\sin(40t)$ into the current model (a) and solve for A, B.

Answers: $A = -\frac{6}{625}$, $B = \frac{8}{625}$.

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(40t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = 0.016$, $\delta = \arctan(0.75)$, $\delta/\omega = 0.0160875$.

References

Course slides on Electric Circuits. Edwards-Penney *Differential Equations and Boundary Value Problems*, section 3.7, course supplement. EP or EPH sections 5.4, 5.5, 5.6.

Quiz 8, Problem 2. Picard's Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let $\vec{f}(x, \vec{y})$ be defined for $|x - x_0| \le h$, $\|\vec{y} - \vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x, \vec{y}(x)), \quad |x - x_0| < H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x - x_0| < H$.

Emile Picard

The Problem. The second order problem

$$(1) \qquad \begin{cases} u'' + 2u' + 17u = 100, \\ u(0) = 1, \\ u'(0) = -1 \end{cases}$$

is a spring-mass model with damping and constant external force. The variables are time x in seconds and elongation $u(x)$ in meters, measured from equilibrium. Coefficients in the equation represent mass $m = 1$ kg, a viscous damping constant $c = 2$, Hooke's constant $k = 17$ and external force $F(x) = 100$.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = -1$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -17y_1 - 2y_2 + 100.$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$(2) \qquad \begin{cases} \vec{y}\,'(x) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\ \vec{y}(0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases}$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

Quiz 8, Problem 3. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y''+16y = 0$ has characteristic equation $r^2+16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
   The solution space is then $S = \mathbf{span}$(the N Euler solution atoms).

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x} \cos(3x), e^{2x} \sin(3x)$.

> **Explained**. The Euler substitution $y = e^{rx}$ produces a solution of the differential equation when r is a complex root of the characteristic equation. Complex exponentials are not used directly. Ever. They are replaced by sines and cosines times real exponentials, which are Euler solution atoms. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies $e^{2x}\cos(3x) = e^{2x}\frac{e^{3xi}+e^{-3xi}}{2} = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$, which is a linear combination of complex exponentials, solutions of the differential equation because of Euler's substitution. Superposition implies $e^{2x}\cos(3x)$ is a solution. Similar for $e^{2x}\sin(3x)$. The independent pair $e^{2x}\cos(3x), e^{2x}\sin(3x)$ replaces both $e^{(2+3i)x}$ and $e^{(2-3i)x}$.

Four complex conjugate roots listed according to multiplicity as $2\pm3i, 2\pm3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x),\ xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a\pm bi$ $(b > 0)$ arising from roots of $(r-a)^2+b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 4y' = 0$

(b) $y'' + 4y = 0$

(c) $y''' + 4y' = 0$

(d) $y'' + 4y = 0,\ y(0) = 1,\ y'(0) = 2$

(e) $y'''' + 81y'' = 0,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r + 1)^2(r^2 - 1) = 0$.

(g) The characteristic equation is $(r - 1)^2(r^2 - 1)^2((r + 1)^2 + 9) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, -1, 2, 2, 3+4i, 3-4i, 3+4i, 3-4i$.