Quiz 7

Quiz7 Problem 1. Independence

The Problem. In the parts below, cite which tests apply to decide on independence or dependence. Choose one test and show complete details.

(a) Vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) Vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(c) Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are data packages constructed from the equations $y = x^2, y = x^5, y = x^{10}$ on $(-\infty, \infty)$.

(d) Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are data packages constructed from the equations $y = 1 + x, y = 1 - x, y = x^3$ on $(-\infty, \infty)$.

Basic Test. To show three vectors are independent, form the system of equations

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0},$$

then solve for c_1, c_2, c_3. If the only solution possible is $c_1 = c_2 = c_3 = 0$, then the vectors are independent.

Linear Combination Test. A list of vectors is independent if and only if each vector in the list is not a linear combination of the remaining vectors, and each vector in the list is not zero.

Subset Test. Any nonvoid subset of an independent set is independent.

The basic test leads to three quick **independence tests for column vectors.** All tests use the augmented matrix A of the vectors, which for 3 vectors is $A =< \vec{v}_1|\vec{v}_2|\vec{v}_3 >$.

Rank Test. The vectors are independent if and only if **rank**(A) equals the number of vectors.

Determinant Test. Assume A is square. The vectors are independent if and only if the determinant of A is nonzero.

Pivot Test. The vectors are independent if and only if all columns of A are pivot columns. The maximum number of independent columns equals the rank, which is the number of pivot columns.

The basic test leads to **independence tests for functions.** Throughout, f_1, f_2, f_3 are given functions to be tested for independence on an interval $a < x < b$. The tests extend in the obvious way to 2 or more functions.

Sampling Test. Invent samples x_1, x_2, x_3 from $a < x < b$ and form the sample matrix $S = \begin{pmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{pmatrix}$. If $|S| \neq 0$, then the functions are independent.

Wronskian Test. Form the Wronskian matrix $W(x) = \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{pmatrix}$.

If $|W(x)| \neq 0$ for some x in $a < x < b$, then the functions are independent.

Euler Solution Atom Test. Any list of distinct Euler Solution Atoms is independent. Atoms are $1, \cos(bx), \sin(bx)$ with $b > 0$, or these three multiplied by e^{ax} with $a \neq 0$, or any of the preceding multiplied by a positive integer power of x, that is, by $x, x^2, x^3, \ldots$.

Quiz7 Problem 2. **Subspaces**

The Problem.

Decide if each of the following sets is a subspace. Cite the theorem that applies (see below), and supply details.

    1. Vector space $V = \mathcal{R}^3$. Set S is defined by relations $x_1 + x_3 = x_2, x_2 + x_3 = 0$.

    2. Vector space $V = \mathcal{R}^3$. Set S is defined by relations $x_3 = x_2, x_1 + x_3 = 1$.

    3. Vector space V of all polynomials $p(x)$. Set S is defined by relations $p(0) = 0, p'(1) = 0$.

    4. Vector space V of all 2×2 matrices A. Set S is defined by $A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

    5. Vector space $V = \mathcal{R}^3$. Set S is all vectors with rational entries.

    6. Vector space $V = \mathcal{R}^3$. Set S is all vectors with non-negative entries.

    7. Vector space $V = \mathcal{R}^3$. Set S is all vectors with $x_1 x_2 x_3 = 1$.

    8. Vector space V of all continuous functions. Set S is all linear combinations of 1 and e^{-10x}.

Subspace Criterion. A subset S of a vector space V is a **subspace** provided the following three items are satisfied.

    (a) The zero vector $\vec{0}$ is in S.

    (b) If $\vec{v}_1, \vec{v}_2$ are in S, then their sum $\vec{v}_1 + \vec{v}_2$ is in S.

    (c) If $\vec{v}$ is in S and c is a scalar, then $c\vec{v}$ is in S.

Kernel Theorem. A subset S of vector space $\mathcal{R}^n$ defined by a system of linear homogeneous algebraic equations is a subspace. Such equations can always be written as $A\vec{x} = \vec{0}$. The solution set of $A\vec{x} = \vec{0}$ is called the **kernel** of matrix A.

Span Theorem. The span of a set of vectors in a vector space V is a subspace of V. This means the set of all linear combinations of a list of vectors is a subspace of V.

Not a Subspace Theorem. A subset S of a vector space V fails to be a subspace if one of the following holds.

    (a) The zero vector $\vec{0}$ is not in S.

    (b) Two vectors $\vec{v}_1, \vec{v}_2$ in S can be defined, for which their sum $\vec{v}_1 + \vec{v}_2$ is not in S.

    (c) A vector $\vec{v}$ in S can be defined for which $-\vec{v}$ is not in S.

A non-homogeneous system of linear equations used to define S can never provide a subspace, because $\vec{x} = \vec{0}$ cannot satisfy $A\vec{x} = \vec{b}$ if $\vec{b} \neq \vec{0}$. For instance, $x + y + z = 1$ in $\mathcal{R}^3$ cannot define a subspace.

Extra Credit Quiz7 Problem 1. **Heat Transfer** and the Mean Value Property.

Credit for this problem will cancel missed credit from any quiz score for the entire semester. The score will not affect homework, lab or exam scores.

Consider the cross section of a long rectangular dam on a river, represented in the figure.
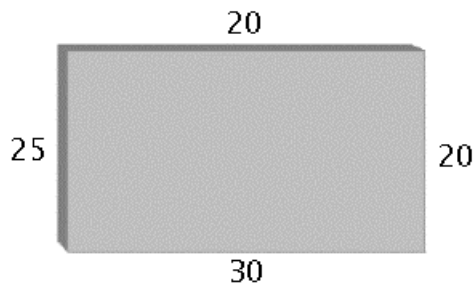
The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.
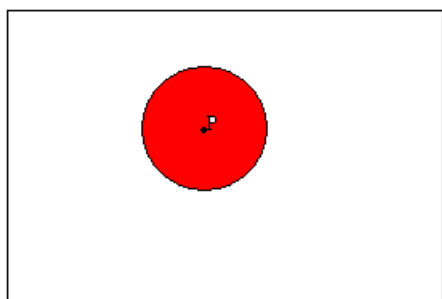
The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle with center P contained in the plate, then the temperature at P is the average value of the temperature function over the boundary of C.

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C. We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4, then use the property to obtain four equations.
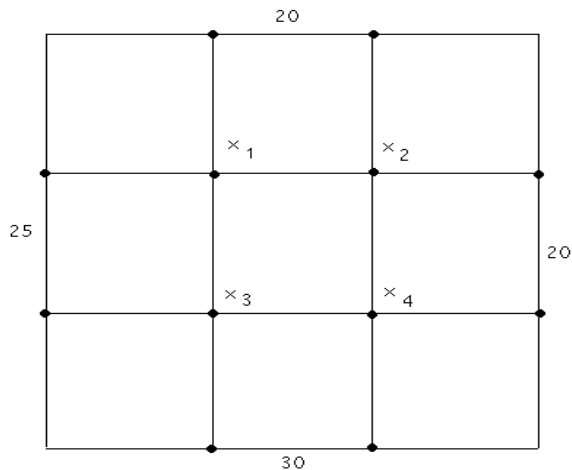
Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

The Problem

(a) Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$. Use technology.

(b) Replace the temperatures $20, 25, 30$ by $20, 16, 12$ and re-compute the four temperatures.

(c) Using the temperatures from part (b), subdivide the grid to make it 6×6. Assign 25 unknown temperatures at the interior grid points. Find the equations and solve using technology.

References. EPH Chapters 12, 13, used for Partial Differential Equations 3150.

Extra Credit Quiz7 Problem 2. Archeology and the Dot Product.

Credit for this problem will cancel missed credit from any quiz score for the entire semester. The score will not affect homework, lab or exam scores.

Archeologist Sir Flinders Petrie collected and analyzed pottery fragments from 900 Egyptian graves. He deduced from the data an historical ordering of the 900 sites. Petrie's ideas will be illustrated for 4 sites and 3 pottery types. The matrix rows below represent sites 1, 2, 3, 4 and the matrix columns represent pottery types 1, 2, 3. A matrix entry is 1 if the site has that pottery type and 0 if not. This is an **incidence matrix**.

$$ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} $$

Petrie Matrix. It is an incidence matrix in which the ones in each column appear together, like the matirx above.

Counting Pottery Types. The dot product of row 2 and row 3 is

$$ \Big(0, 1, 1 \Big) \cdot \Big(1, 0, 1 \Big) = 0 * 1 + 1 * 0 + 1 * 1 = 1 $$

which means sites 2 and 3 have one pottery type in common. **Please pause on this arithmetic**, until you agree that the products $0 * 1, 1 * 0, 1 * 1$ add to the number of pottery types in common.

Sites with pottery in common are expected to be historically close in time. Because pottery types evolve, old types cease production when newly created pottery types begin production, which gives meaning to the clustered ones in the columns of A.

The Problem. Find a sequence of row swaps which starts with the incidence matrix

$$ C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} $$

and ends with a Petrie matrix A. Express the swaps as elementary matrices $E_1, E_2, \ldots$ and write A as the product of elementary matrices times C. There is not a unique set of swaps, because of duplicate rows.

You may use Kendall's ideas with Robinson matrices or the simple definition of a Petrie matrix, given above. In any case, after finding Petrie matrix A, compute Robinson's matrix $R = AA^T$ and apply the ideas in sample quiz 7, to double-check the answer.

References. Edwards-Penney Sections 3.4, 3.5, 3.6. David Kendall's 1969 work, Incidence matrices, interval graphs and seriation in archeology. W. S. Robinson, the method for chronologically ordering archeological deposits, in the April 1951 issue of American Antiquity. The incidence matrix C was invented from an archeology exercise on Seriation of 7 Maryland sites based on 3 types of historical ceramics.