Quiz 12

Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a short trip from Salt Lake City to Evanston, on the Wyoming border. The route is I-80 E, 75 miles through Utah. Google maps estimates 1 hour and 11 minutes driving time. The table below shows the distances, time, road segment and average speed with total trip time 1 hour and 38 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.5	Kimball to Wanship	65 – 55
9.1	11	49.6	Wanship to Coalville	70
5.7	6	57	Coalville to Echo Dam	70
16.5	16	61.9	Echo Dam to 75 mph sign	70
39	33	70.9	75 mph sign to Evanston	75

The **velocity function** for the SUV is approximated by

$$V_{\text{pc}}(t) = \begin{cases} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\ \hline 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.5 & 20 < t < 32 & \text{Kimball to Wanship} \\ 49.6 & 32 < t < 43 & \text{Wanship to Coalville} \\ 57.0 & 43 < t < 49 & \text{Coalville to Echo Dam} \\ 61.9 & 49 < t < 65 & \text{Echo Dam to 75 mph sign} \\ 70.1 & 65 < t < 98 & \text{75 mph sign to Evanston} \end{cases}$$

The velocity function $V_{\text{pc}}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \begin{cases} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{cases}$$

where functions $f_1, f_2, \ldots, f_n$ are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(0) = \lim_{h \to 0+} f(h)$, which allows the parts rule $\mathcal{L}(f'(t)) = s\,\mathcal{L}(f(t)) - f(0)$.

The Problem. The SUV travels from $t = 0$ to $t = \frac{98}{60} = 1.6$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{\text{pc}}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{\text{pc}}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\text{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Problem 2. Switches and Impulses

Laplace's method solves differential equations. It is the premier method for solving equations containing switches or impulses.

Unit Step Define $u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$ It is a **switch**, turned on at $t = a$.

Ramp Define **ramp**$(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \ge a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse**$(t, a, b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \le t \le b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b-a}\right)(b-a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t-a) = \frac{du}{dt}(t-a)$, where $u(t-a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Steiltjes integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t-a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t-a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\,\textbf{pulse}(t, a-h, a+h).$$
- The fundamental equation is $\int_{-\infty}^{\infty} F(x)\delta(x-a)dx = F(a)$. Symbol $\delta(t-a)$ is not manipulated as an ordinary function, but regarded as $du(t-a)/dt$ in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \ge 0$,

Forward table	**Backward table**
$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\,\mathcal{L}(f(t))$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\,\mathcal{L}\left(g(t)\vert_{t=t+a}\right)$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)\vert_{t=t-a}\right)$.

The Problem. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for terms involving the unit step, ramp and pulse, in these special cases:

$$(1)\ \mathcal{L}((t-1)u(t-1)) \quad (2)\ \mathcal{L}(e^t\,\textbf{ramp}(t-2)), \quad (3)\ \mathcal{L}(5\,\textbf{pulse}(t,2,4)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1)\ \mathcal{L}(f) = \frac{e^{-2s}}{s} \quad (2)\ \mathcal{L}(f) = \frac{e^{-s}}{(s+1)^2} \quad (3)\ \mathcal{L}(f) = e^{-s}\frac{3}{s} - e^{-2s}\frac{3}{s}.$$

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

$$(1)\ \mathcal{L}(2\delta(t-5)), \quad (2)\ \mathcal{L}(2\delta(t-1)+5\delta(t-3)), \quad (3)\ \mathcal{L}(e^t\delta(t-2)).$$

The sum of Dirac impulses in (2) is called an **impulse train**. The numbers 2 and 5 represent the applied **impulse** at times 1 and 3, respectively.

Reference: The Riemann-Stieltjes Integral

Definition

The Riemann-Stieltjes integral of a real-valued function f of a real variable with respect to a real monotone non-decreasing function g is denoted by

$$\int_a^b f(x)\,dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating RiemannStieltjes sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the **integrand** and the **integrator**.

The **limit** is a number A, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\mathbf{mesh}(P) = \max_{0 \le i < n} (x_{i+1} - x_i) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$|S(P, f, g) - A| < \varepsilon.$$