Matrix Exponential: Putzer Formula
Variation of Parameters for Systems
Undetermined Coefficients for Systems

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The 2 × 2 Matrix Exponential e^{At}

The matrix e^{At} has columns equal to the solutions of the two problems

$$\begin{cases} \vec{u}_1'(t) = A\vec{u}_1(t), \\ \vec{u}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \qquad \begin{cases} \vec{u}_2'(t) = A\vec{u}_2(t), \\ \vec{u}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Briefly, the matrix $\Phi(t) = e^{At}$ satisfies the two conditions

1. $\Phi'(t) = A\Phi(t)$,
2. $\Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Putzer Formula for 2×2 Matrices

$$e^{\boldsymbol{A}t} = e^{\lambda_1 t}\boldsymbol{I} + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$$
$\boldsymbol{A}$ is 2×2, $\lambda_1 \neq \lambda_2$ real.

$$e^{\boldsymbol{A}t} = e^{\lambda_1 t}\boldsymbol{I} + t e^{\lambda_1 t}(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$$
$\boldsymbol{A}$ is 2×2, $\lambda_1 = \lambda_2$ real.

$$e^{\boldsymbol{A}t} = e^{at}\cos bt\, \boldsymbol{I} + \frac{e^{at}\sin bt}{b}(\boldsymbol{A} - a\boldsymbol{I})$$
$\boldsymbol{A}$ is 2×2, $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$.

How to Remember Putzer's 2×2 Formula

The expressions

$$
\begin{aligned}
e^{At} &= r_1(t)I + r_2(t)(A - \lambda_1 I), \\
r_1(t) &= e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{aligned}
\tag{1}
$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton quotient for r_1. It has limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}.$$

Taking real parts of expression (1) gives the complex case formula.

Variation of Parameters

Theorem 1 (Variation of Parameters for Systems)
Let A be a constant $n \times n$ matrix and $\mathbf{F}(t)$ a continuous function near $t = t_0$. The unique solution $\mathbf{x}(t)$ of the matrix initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{F}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

is given by the **variation of parameters formula**

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_{t_0}^{t} e^{-rA}\mathbf{F}(r)dr. \tag{2}$$

Theorem 2 (Polynomial solutions)
Let $\boldsymbol{f}(\boldsymbol{t})$ be a polynomial of degree $\boldsymbol{k}$. Assume $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ constant invertible matrix. Then $\mathbf{u}' = \boldsymbol{A}\mathbf{u} + \boldsymbol{f}(\boldsymbol{t})\mathbf{c}$ has a polynomial solution $\mathbf{u}(\boldsymbol{t}) = \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ of degree $\boldsymbol{k}$ with vector coefficients $\{\mathbf{c}_j\}$ given by the relations

$$\mathbf{c}_j = -\sum_{i=j}^{k} \boldsymbol{f}^{(i)}(0)\boldsymbol{A}^{j-i-1}\mathbf{c}, \quad 0 \leq j \leq k.$$