2.9 Exact Equations and Level Curves

A **level curve** or a **conservation law** is an equation of the form

$$U(x, y) = c.$$

Hikers like to think of U as the *altitude* at position (x, y) on the map and $U(x, y) = c$ as the *curve* which represents the easiest walking path, that is, altitude does not change along that route. The altitude is **conserved** along the route, hence the terminology *conservation law*.

Other examples of level curves are *isobars* and *isotherms*. An **isobar** is a planar curve where the atmospheric pressure is constant. An **isotherm** is a planar curve along which the temperature is constant.

Definition 3 (Potential)
The function $U(x, y)$ in a conservation law is called a **potential**. The **dynamical equation** is the first order differential equation

$$(1) \qquad M dx + N dy = 0, \quad M = U_x(x, y), \quad N = U_y(x, y).$$

The *dynamics* or *changes* in x and y are described by (1). To **solve** $M dx + N dy = 0$ means this: find a conservation law $U(x, y) = c$ so that (1) holds. Formally, (1) is found by *implicit differentiation* of $U(x, y) = c$; see *Technical Details*, page 154.

The Potential Problem and Exactness

The **potential problem** assumes given a dynamical equation $M dx + N dy = 0$ and seeks to find a potential $U(x, y)$ from the set of equations

$$(2) \qquad \begin{aligned} U_x &= M(x, y), \\ U_y &= N(x, y). \end{aligned}$$

If some potential $U(x, y)$ satisfies equation (2), then $M dx + N dy = 0$ is said to be **exact**. It is a consequence of the mixed partial equality $U_{xy} = U_{yx}$ that the existence of a solution U implies $M_y = N_x$. Surprisingly, this condition is also sufficient.

Theorem 5 (Exactness)
Let $M(x, y)$, $N(x, y)$ and their first partials be continuous in a rectangle D. Assume $M_y(x, y) = N_x(x, y)$ in D and (x_0, y_0) is a point of D. Then the equation $M dx + N dy = 0$ is exact with potential U given by the formula

$$(3) \qquad U(x, y) = \int_{x_0}^{x} M(t, y) dt + \int_{y_0}^{y} N(x_0, s) ds.$$

The proof is delayed to page 154.

The Method of Potentials

Formula (3) has technical problems because it requires two integrations. The integrands have a *parameter*: they are *parametric integrals*. Integration effort can be reduced by using the **method of potentials** for $Mdx + Ndy = 0$, which applies equation (3) with $x_0 = y_0 = 0$ in order to simplify integrations.

Test $M_y = N_x$	Compute the partials M_y and N_x, then test the equality $M_y = N_x$. Proceed if equality holds.
Trial Potential	Let $U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$. Evaluate both integrals.
Test $U(x,y)$	Compute U_x and U_y, then test both $U_x = M$ and $U_y = N$. This step finds integration errors.

Examples

40 **Example (Exactness Test)** Test $Mdx + Ndy = 0$ for the existence of a potential U, given $M = 2xy + y^3 + y$ and $N = x^2 + 3xy^2 + x$,

Solution: Theorem 5 implies that $Mdx + Ndy = 0$ has a potential U exactly when $M_y = N_x$. It suffices to compute the partials and show they are equal.

$$M_y = \partial_y(2xy + y^3 + y) = 2x + 3y^2 + 1, \qquad N_x = \partial_x(x^2 + 3xy^2 + x) = 2x + 3y^2 + 1.$$

41 **Example (Conservation Law Test)** Test whether $U = x^2y + xy^3 + xy$ is a potential for $Mdx + Ndy = 0$, given $M = 2xy + y^3 + y$, $N = x^2 + 3xy^2 + x$.

Solution: By definition, it suffices to test the equalities $U_x = M$ and $U_y = N$.

$$U_x = \partial_x(x^2y + xy^3 + xy) = 2xy + y^3 + y = M, \qquad U_y = \partial_y(x^2y + xy^3 + xy) = x^2 + 3xy^2 + x = N.$$

42 **Example (Method of Potentials)** Solve $y' = -\dfrac{2xy + y^3 + y}{x^2 + 3xy^2 + x}$.

Solution: The implicit solution $x^2y + xy^3 + xy = c$ will be justified.

The equation has the form $Mdx + Ndy = 0$ where $M = 2xy + y^3 + y$ and $N = x^2 + 3xy^2 + x$. It is exact, by Theorem 5, because the partials $M_y = 2x + 3y^2 + 1$ and $N_x = 2x + 3y^2 + 1$ are equal.

The method of potentials applies to find the potential $U = x^2y + xy^3 + xy$ as follows.

$U = \int_0^x M(x,y)dx + \int_0^y N(0,y)dy$ Formula for U, Theorem 5.

$= \int_0^x \left(2xy + y^3 + y\right) dx + \int_0^y (0) dy$	Insert M and N.
$= x^2y + xy^3 + xy$	Evaluate integral.

Observe that $N(x, y)$ simplifies to zero at $x = 0$, which reduces the actual work in half. Any choice other than $x_0 = 0$ in Theorem 5 increases the labor.

To *test the solution*, compute the partials of U, then compare them to M and N; see Example 41.

43 Example (Exact Equation) Solve $\dfrac{x + y}{(1 - x)^2} dx + \dfrac{x}{1 - x} dy = 0$.

Solution: The implicit solution $\dfrac{xy + x}{1 - x} + \ln|x - 1| = c$ will be justified.

Assume given the exactness of the equation $Mdx + Ndy = 0$, where $M = (x + y)/(1 - x)^2$ and $N = x/(1 - x)$. Apply Theorem 5:

$U = \int_0^x M(x, y) dx + \int_0^y N(0, y) dy$	Method of potentials.
$= \int_0^x \dfrac{x + y}{(1 - x)^2} dx + \int_0^y (0) dy$	Substitute for M, N.
$= \int_0^x \left(\dfrac{y + 1}{(x - 1)^2} + \dfrac{1}{x - 1}\right) dx$	Partial fractions.
$= \dfrac{xy + x}{1 - x} + \ln|x - 1|$	Evaluate integral.

Additional examples, including the context for the preceding example, appear in the next section.

Remarks on the Method of Potentials

Indefinite integrals $\int M(x, y) dx$ and $\int N(0, y) dy$ can be used, provided the two integration answers are zero at $x = 0$ and $y = 0$, respectively. Swapping the roles of x and y gives $U = \int_0^y N(x, y) dy + \int_0^x M(x, 0) dx$, a form which may have easier integrations.

Can the test $M_y = N_x$ be skipped? True, it is enough to verify that the potential works (the last step). If the last step fails, then the first step must be done to resolve the error.

The equation $ydx + 2xdy = 0$ fails $M_y = N_x$ and the trial potential $U = xy$ fails $U_x = M$, $U_y = N$. In the equivalent form $x^{-1}dx + 2y^{-1}dy = 0$, the method of potentials does not apply directly, because $(0, 0)$ is outside the domain of continuity. Nevertheless, the trial potential $U = \ln x + 2\ln y$ passes the test $U_x = M$, $U_y = N$. Such pleasant accidents account for the popularity of the method of potentials.

It is prudent in applications of Theorem 5 to test $x_0 = y_0 = 0$ in M and N, to detect a discontinuity. If detected, then another vertex x_0, y_0 of the unit square, e.g., $x_0 = y_0 = 1$, might suffice.

Details and Proofs

Justification of equation (1) uses the calculus *chain rule*

$$\frac{d}{dt}U(x(t), y(t)) = U_x(x(t), y(t))x'(t) + U_y(x(t), y(t))y'(t)$$

and differential notation $dx = x'(t)dt$, $dy = y'(t)dt$. To justify (1), let $(x(t), y(t))$ be some parameterization of the level curve, then differentiate on t across the equation $U(x(t), y(t)) = c$ and apply the chain rule.

Proof of Theorem 5

Background result. The proof assumes the following identity:

$$\frac{\partial}{\partial y}\int_{x_0}^{x} M(t, y)dt = \int_{x_0}^{x} M_y(t, y)dt.$$

The identity is obtained by forming the Newton quotient $(G(y+h)-G(y))/h$ for the derivative of $G(y) = \int_{x_0}^{x} M(t, y)dt$ and then taking the limit as h approaches zero. Technically, the limit must be taken inside an integral sign, which for success requires continuity of the partial M_y.

Details. It has to be shown that the implicit relation $U(x, y) = c$ with U defined by (3) is a solution of the exact equation $Mdx + Ndy = 0$, that is, the relations $U_x = M$, $U_y = N$ hold. The partials are calculated from the background result as follows.

$U_x = \partial_x \int_{x_0}^{x} M(t, y)dt$	Use (3), in which the second integral does not depend on x.
$= M(x, y),$	Fundamental theorem of calculus.
$U_y = \partial_y \int_{x_0}^{x} M(t, y)dt$ $+ \partial_y \int_{y_0}^{y} N(x_0, s)ds$	Use (3).
$= \int_{x_0}^{x} M_y(t, y)dt + N(x_0, y)$	Apply the background result and the fundamental theorem.
$= \int_{x_0}^{x} N_x(t, y)dt + N(x_0, y)$	Substitute $M_y = N_x$.
$= N(x, y)$	Fundamental theorem of calculus.

The verification is complete.

Power Series Proof of Theorem 5 It will be assumed that M and N have power series expansions about $x = y = 0$. Let $U_1 = \int M(x, y)dx$ and $U_2 = \int N(x, y)dy$ with $U_1(0, y) = U_2(x, 0) = 0$. The series forms of U_1 and U_2 will be

$$U_1 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}x^i y^j + \sum_{i=1}^{\infty} a_i x^i,$$

$$U_2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} d_{ij}x^i y^j + \sum_{j=1}^{\infty} b_j y^j.$$

The identities $\partial_y\partial_x U_1 = M_y = N_x = \partial_x\partial_y U_2$ imply that $c_{ij} = d_{ij}$, using term-by-term differentiation. The trial potential is $U = U_1 + \sum_{j=1}^{\infty} b_j y^j$ or $U = U_2 + \sum_{i=1}^{\infty} a_i x^i$. From these relations it follows that $U_x = M$ and $U_y = N$. Therefore, $Mdx + Ndy = 0$ is exact with potential U.

A Popular Method. The power series proof justifies this method: *the potential U is the sum of $\int Mdx$ and the terms from $\int Ndy$ which do not appear in $\int Mdx$.*

Simplifications of the integrand in $\int N(0, y)dy$, due to $x = 0$, suggest that $\int N(x, y)dy$ might involve more labor. Examples show that this insight is correct.

Exercises 2.9

Exactness Test. Test the equality $M_y = N_x$ for the given equation, as written, and report *exact* when true. Do not try to solve the differential equation. See Example 40, page 152.

1. $(y - x)dx + (y + x)dy = 0$

2. $(y + x)dx + (x - y)dy = 0$

3. $(y + \sqrt{xy})dx + (-y)dy = 0$

4. $(y + \sqrt{xy})dx + xydy = 0$

5. $(x^2 + 3y^2)dx + 6xydy = 0$

6. $(y^2 + 3x^2)dx + 2xydy = 0$

7. $(y^3 + x^3)dx + 3xy^2dy = 0$

8. $(y^3 + x^3)dx + 2xy^2dy = 0$

9. $2xydx + (x^2 - y^2)dy = 0$

10. $2xydx + (x^2 + y^2)dy = 0$

Conservation Law Test. For each given conservation law $U(x, y) = c$, report whether or not it is a solution to $Mdx + Ndy = 0$. See Example 41, page 152.

11. $2xydx + (x^2 + 3y^2)dy = 0$, $x^2y + y^3 = c$

12. $2xydx + (x^2 - 3y^2)dy = 0$, $x^2y - y^3 = c$

13. $(3x^2 + 3y^2)dx + 6xydy = 0$, $x^3 + 3xy^2 = c$

14. $(x^2 + 3y^2)dx + 6xydy = 0$, $x^3 + 3xy^2 = c$

15. $(y - 2x)dx + (2y + x)dy = 0$, $xy - x^2 + y^2 = c$

16. $(y + 2x)dx + (-2y + x)dy = 0$, $xy + x^2 - y^2 = c$

Exactness Theorem. Apply the exactness Theorem 5 and possibly the method of potentials to find an implicit solution $U(x, y) = c$ for the given differential equation. See Examples 42-43, page 152.

17. $(y - 4x)dx + (4y + x)dy = 0$

18. $(y + 4x)dx + (4y + x)dy = 0$

19. $(e^y + e^x)dx + (xe^y)dy = 0$

20. $(e^{2y} + e^x)dx + (2xe^{2y})dy = 0$

21. $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$

22. $(1+ye^{-xy})dx+(xe^{-xy}-4y)dy = 0$

23. $(2x + \arctan y)dx + \dfrac{x}{1 + y^2}\, dy = 0$

24. $(2x + \arctan y)dx + \dfrac{x + 2y}{1 + y^2}\, dy = 0$

25. $\dfrac{2x^5 + 3y^3}{x^4 y}dx - \dfrac{2y^3 + x^5}{x^3 y^2}dy = 0$

26. $\dfrac{2x^4 + y^2}{x^3 y}dx - \dfrac{2x^4 + y^2}{2x^2 y^2}dy = 0$

27. $Mdx + Ndy = 0$, $M = e^x \sin y + \tan y$, $N = e^x \cos y + x \sec^2 y$

28. $Mdx + Ndy = 0$, $M = e^x \cos y + \tan y$, $N = -e^x \sin y + x \sec^2 y$

29. $\left(x^2 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

30. $\left(x^3 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

2.10 Special equations

Homogeneous-A Equation

A first order equation of the form $y' = F(y/x)$ is called a **homogeneous class A equation**. . The substitution $u = y/x$ changes it into an equivalent first order separable equation $xu' + u = F(u)$. Solutions of $y' = F(y/x)$ and $xu' + u = F(u)$ are related by the relation $y = xu$.

Homogeneous-C Equation

Let $R(x, y)$ be a rational function constructed from *two affine functions*:

$$R(x,y) = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}.$$

A first order equation of the form $y' = G(R(x, y))$ is called a **homogeneous class C equation** . If the system

$$a_1a + b_1b = c_1, \quad a_2a + b_2b = c_2$$

has a solution (a, b), then the change of variables $x = X - a$, $y = Y - b$ effectively eliminates the terms c_1 and c_2. Accordingly, the equation $y' = G(R(x, y))$ converts into a homogeneous class A equation

$$Y' = G\left(\frac{a_1 + b_1Y/X}{a_2 + b_2Y/X}\right).$$

This equation type was solved in the previous paragraph. Justification follows from $y' = Y'$ and $R(X - a, Y - b) = (a_1X + b_1Y)/(a_2X + b_2Y)$.

Bernoulli's Equation

The equation $y' + p(x)y = q(x)y^n$ is called the **Bernoulli differential equation**. If $n = 1$ or $n = 0$, then this is a linear equation. Otherwise, the substitution $u = y/y^n$ changes it into the linear first order equation $u' + (1 - n)p(x)u = (1 - n)q(x)$.

Integrating Factors and Exact Equations

An equation $\mathbf{M}dx + \mathbf{N}dy = 0$ is said to have an **integrating factor** $Q(x, y)$ if multiplication across the equation by Q produces an exact equation $Mdx + Ndy = 0$. The definition implies $M = Q\mathbf{M}$, $N = Q\mathbf{N}$ and $M_y = N_x$. The search for Q is only interesting when $\mathbf{M}_y \ne \mathbf{N}_x$.

A systematic approach to finding Q includes a list of **trial integrating factors**, which are known to work for special equations:

$Q = x^a y^b$	Require $xy\left(\mathbf{M}_y - \mathbf{N}_x\right) = ay\mathbf{N} - bx\mathbf{M}$. This integrating factor can introduce *extraneous solutions* $x = 0$ or $y = 0$.
$Q = e^{ax+by}$	Require $\mathbf{M}_y - \mathbf{N}_x = a\mathbf{N} - b\mathbf{M}$.
$Q = e^{\int \mu(x)dx}$	Require $\mu = \left(\mathbf{M}_y - \mathbf{N}_x\right)/N$ to be independent of y.
$Q = e^{\int \nu(y)dy}$	Require $\nu = \left(\mathbf{N}_x - \mathbf{M}_y\right)/M$ to be independent of x.

Examples

44 **Example (Homogeneous-A)** Solve $yy' = 2x + y^2/x$

Solution: The *implicit solution* will be shown to be

$$y^2 = cx^2 + 4x^2 \ln x.$$

The equation $yy' = 2x + y^2/x$ is not separable, linear nor exact. Division by y gives the homogeneous-A form $y' = 2/u + u$ where $u = y/x$. Then

$xu' + u = 2/u + u$	Form $xu' + u = F(u)$.
$xu' = 2/u$	Separable form.
$u^2 = c + 4\ln x$	Implicit solution u.
$y^2 = x^2u^2$	Change of variables $y = xu$.
$\quad = cx^2 + 4x^2 \ln x$	Substitute $u^2 = c + 4\ln x$.

Check the implicit solution against $yy' = 2x + y^2/x$ as follows.

$\text{LHS} = yy'$	Left side of $yy' = 2x + y^2/x$.
$\quad = \frac{1}{2}(y^2)'$	Calculus identity.
$\quad = \frac{1}{2}(cx^2 + 4x^2 \ln x)'$	Substitute.
$\quad = cx + 4x \ln x + 2x$	Differentiate.
$\quad = 2x + y^2/x$	Use $y^2 = cx^2 + 4x^2 \ln x$.
$\quad = \text{RHS}$.	Equality verified.

45 **Example (Homogeneous-C)** Solve $y' = \dfrac{x+y+3}{x-y+5}$.

Solution: The *implicit solution* will be shown to be

$$2\ln(x+4) + \ln\left(\left(\frac{y-1}{x+4}\right)^2 + 1\right) - 2\arctan\left(\frac{y-1}{x+4}\right) = c.$$

The equation would be of type homogeneous-A, if not for the constants 3 and 5 in the fraction $(x+y+3)/(x-y+5)$. The method applies a translation of coordinates $x = X - a$, $y = Y - b$ as below.

$x + y + 3 = X + Y,$ $x - y + 5 = X - Y$	Require the translation to remove the constant terms.

$3 = a+b,$ $5 = a-b$	Substitute $X = x+a$, $Y = y+b$ and simplify.
$a = 4,\ b = -1$	Unique solution of the system.
$\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y}$	Translated type homogeneous-A equation.
$X\dfrac{du}{dX} + u = \dfrac{1+u}{1-u}$	Use $u = Y/X$ to eliminate Y.
$\dfrac{1-u}{1+u^2}\dfrac{du}{dX} = \dfrac{1}{X}$	Separated form.

The separated form is integrated as $\int du/(1+u^2) - \int udu/(1+u^2) = \int dX/X$. Evaluation gives the implicit solution

$$\arctan(u) - \frac{1}{2}\ln\left(u^2+1\right) = C + \ln X.$$

Changing variables $x = X - 4$, $y = Y + 1$ and consolidating constants produces the announced solution.

To check the solution by `maple` assist, use the following code, which tests $U(x, y) = c$ against $y' = f(x, y)$. The test succeeds if `odetest` returns zero.

```
# Maple V 5.1
U:=(x,y)->2*ln(x+4)+ln(((y-1)/(x+4))^2+1)-2*arctan((y-1)/(x+4));
f:=(x,y)->(x+y+3)/(x-y+5); DE:=diff(y(x),x)=f(x,y(x));
odetest(U(x,y(x))=c,DE);
```

46 Example (Bernoulli) Solve $y' + 2y = y^2$.

Solution: It will be shown that the solution is $y = \dfrac{1}{1+Ce^x}$.

The equation can be solved by other methods, notably separation of variables. Bernoulli's substitution $u = y/y^n$ will be applied to find the equivalent first order linear differential equation, as follows.

$u' = (y/y^2)'$	Bernoulli's substitution, $n = 2$.
$= -y^{-2}y'$	Chain rule.
$= -1 + y^{-1}$	Use $y' + 2y = y^2$.
$= -1 + u$	Use $u = y/y^2$.

This linear equation $u' = -1 + u$ has equilibrium solution $u_p = 1$ and homogeneous solution $u_h = Ce^x$. Therefore, $u = u_h + u_p$ gives $y = u^{-1} = 1/(1+Ce^x)$.

47 Example ($Q = x^a y^b$) Solve $(3y + 4xy^2)dx + (4x + 5x^2y)dy = 0$.

Solution: The implicit solution $x^3y^4 + x^4y^5 = c$ will be justified.

The equation is not exact as written. To explain why, let $\mathbf{M} = 3y + 4xy^2$ and $\mathbf{N} = 4x + 5x^2y$. Then $\mathbf{M}_y = 8xy + 3$, $\mathbf{N}_x = 10xy + 4$ which implies $\mathbf{M}_y \neq \mathbf{N}_x$ (not exact).

The factor $Q = x^a y^b$ will be an integrating factor for the equation provided a and b are chosen to satisfy $xy\,(\mathbf{M}_y - \mathbf{N}_x) = ay\mathbf{N} - bx\mathbf{M}$. This requirement becomes $xy\,(-2xy-1) = ay(4x+5x^2y) - bx(3y+4xy^2)$. Comparing terms across the equation gives the 2×2 system of equations

$$\begin{array}{rcrcr} 4a & - & 3b & = & -1, \\ 5a & - & 4b & = & -2. \end{array}$$

The unique solution by Cramer's determinant rule is

$$a = \frac{\begin{vmatrix} -1 & -3 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix}} = 2, \quad b = \frac{\begin{vmatrix} 4 & -1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix}} = 3.$$

Then $Q = x^2y^3$ is the required integrating factor. After multiplication by Q, the original equation becomes the exact equation

$$(3x^2y^4 + 4x^3y^5)dx + (4x^3y^3 + 5x^4y^4)dy = 0.$$

The method of potentials applied to $M = 3x^2y^4 + 4x^3y^5$ and $N = 4x^3y^3 + 5x^4y^4$ finds the potential U as follows.

$$\begin{aligned} U &= \textstyle\int_0^x M(x,y)dx + \int_0^y N(0,y)dy && \text{Method of potentials formula.} \\ &= \textstyle\int_0^x (3x^2y^4 + 4x^3y^5)dx + \int_0^y (0)dy && \text{Insert } M \text{ and } N. \\ &= x^3y^4 + x^4y^5 && \text{Evaluate integral.} \end{aligned}$$

48 Example ($Q = e^{ax+by}$) Solve $(e^x + e^y)\,dx + (e^x + 2e^y)\,dy = 0$.

Solution: The implicit solution $2e^{3x+3y} + 3e^{2x+4y} = c$ will be justified. A constant 5/6 appears in the integrations below, mysteriously absent in the solution, because 5/6 has been absorbed into the constant c.

Let $\mathbf{M} = e^x + e^y$ and $\mathbf{N} = e^x + 2e^y$. Then $\mathbf{M}_y = e^y$ and $\mathbf{N}_x = e^x$ (not exact). The condition for $Q = e^{ax+by}$ to be an integrating factor is $\mathbf{M}_y - \mathbf{N}_x = a\mathbf{N} - b\mathbf{M}$, which becomes the requirement

$$e^y - e^x = a\,(e^x + 2e^y) - b\,(e^x + e^y)\,.$$

The equations are satisfied provided (a,b) is a solution of the 2×2 system of equations

$$\begin{array}{rcrcr} a & - & b & = & -1, \\ 2a & - & b & = & 1. \end{array}$$

The unique solution is $a = 2$, $b = 3$, by elimination. The original equation multiplied by the integrating factor $Q = e^{2x+3y}$ is the exact equation $Mdx + Ndy = 0$, where $M = e^{3x+3y} + e^{2x+4y}$ and $N = e^{3x+3y} + 2e^{2x+4y}$. The method of potentials applies to find the potential U, as follows.

$$\begin{aligned} U &= \textstyle\int_0^x M(x,y)dx + \int_0^y N(0,y)dy && \text{Method of potentials.} \\ &= \textstyle\int_0^x \left(e^{3x+3y} + e^{2x+4y}\right)dx + \int_0^y \left(e^{3y} + 2e^{4y}\right)dy && \text{Insert } M \text{ and } N. \\ &= \tfrac{1}{3}e^{3x+3y} + \tfrac{1}{2}e^{2x+4y} - \tfrac{5}{6} && \text{Evaluate integral.} \end{aligned}$$

49 Example ($Q = Q(x)$) Solve $(x + y)dx + (x - x^2)dy = 0$.

Solution: The implicit solution $\dfrac{xy + x}{1 - x} + \ln|x - 1| = c$ will be justified.

Let $\mathbf{M} = x + y$, $\mathbf{N} = x - x^2$. Then $\mathbf{M}_y = 1$ and $\mathbf{N}_x = 1 - 2x$ (not exact). Then

$\mu = \dfrac{\mathbf{M}_y - \mathbf{N}_x}{\mathbf{N}}$	Hope μ depends on x alone.
$= 2/(1 - x)$	Substitute $\mathbf{M}$, $\mathbf{N}$; success.
$Q = e^{\int \mu(x)dx}$	Integrating factor.
$= e^{-2\ln\|1-x\|}$	Substitute for μ and integrate.
$= (1 - x)^{-2}$	Simplified factor found.

Multiplication of $\mathbf{M}dx + \mathbf{N}dy = 0$ by Q gives the corresponding exact equation

$$\frac{x + y}{(1 - x)^2}dx + \frac{x}{1 - x}dy = 0.$$

The method of potentials applied to $M = (x + y)/(1 - x)^2$, $N = x/(1 - x)$ finds the implicit solution as follows.

$U = \int_0^x M(x, y)dx + \int_0^y N(0, y)dy$	Method of potentials.
$= \int_0^x \dfrac{x + y}{(1 - x)^2}dx + \int_0^y (0)dy$	Substitute for M, N.
$= \int_0^x \left(\dfrac{y + 1}{(x - 1)^2} + \dfrac{1}{x - 1}\right)dx$	Partial fractions.
$= \dfrac{xy + x}{1 - x} + \ln\|x - 1\|$	Evaluate integral.

50 Example ($Q = Q(y)$) Solve $(y - y^2)dx + (x + y)dy = 0$.

Solution: Interchange the roles of x and y, then apply the previous example, to obtain the implicit solution $\dfrac{xy + y}{1 - y} + \ln|y - 1| = c$.

This example happens to fit the case when the integrating factor is a function of y alone. The details parallel the previous example.

Details and Proofs

The exactness condition $M_y = N_x$ for $M = Q\mathbf{M}$ and $N = Q\mathbf{N}$ becomes in the case $Q = x^a y^b$ the relation

$$bx^a y^{b-1}\mathbf{M} + x^a y^b \mathbf{M}_y = ax^{a-1}y^b\mathbf{N} + x^a y^b \mathbf{N}_x$$

from which rearrangement gives $xy\left(\mathbf{M}_y - \mathbf{N}_x\right) = ay\mathbf{N} - bx\mathbf{M}$. The case $Q = e^{ax+by}$ is similar.

Consider $Q = e^{\int \mu(x)dx}$. Then $Q' = \mu Q$. The exactness condition $M_y = N_x$ for $M = Q\mathbf{M}$ and $N = Q\mathbf{N}$ becomes $Q\mathbf{M}_y = \mu Q\mathbf{N} + Q\mathbf{N}_x$ and finally

$$\mu = \frac{\mathbf{M}_y - \mathbf{N}_x}{\mathbf{N}}.$$

The similar case $Q = e^{\int \nu(y)dy}$ is obtained from the preceding case, by swapping the roles of x, y.

Exercises 2.10

Homogeneous-A Equations. Find f such that the equation can be written in the form $y' = f(y/x)$, then solve for y. Check the answer using a computer algebra system.

1. $xy' = y^2/x$

2. $x^2y' = x^2 + y^2$

3. $yy' = \frac{xy^2}{x^2+y^2}$

4. $yy' = \frac{2xy^2}{4x^2+y^2}$

5. $y' = \frac{y^2}{4x^2+y^2}$

6. $y' = \frac{y^2}{x^2+y^2}$

7. $y' = \frac{y^2}{(x+y)^2}$

8. $y' = \frac{xy}{(x+y)^2}$

9. $y' = \frac{y\left(y^2+4\,yx+5\,x^2\right)}{x(y+2\,x)^2}$

10. $y' = \frac{y^2(y+2\,x)}{x(y+x)^2}$

Homogeneous-C Equations. Decompose $f = G(R(x,y))$ where $R(x,y) = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, then solve $y' = f(x,y)$.

11. $y' = \frac{(y+1)x}{y^2+2\,y+1+x^2}$

12. $y' = 2\,\frac{(y+1)x}{4\,y^2+8\,y+4+x^2}$

13. $y' = \frac{(x+1)^2}{4\,y^2+x^2+2\,x+1}$

14. $y' = \frac{(x+1)^2}{(x+1+y)^2}$

15. $y' = \frac{(y+x)(x+1)}{(2\,x+1+y)^2}$

16. $y' = \frac{x\left(2\,y^2+6\,yx+5\,x^2\right)}{(y+x)(y+2\,x)^2}$

17. $y' = \frac{(y+x)\left(3\,y^2+6\,yx+2\,y+3\,x^2+2\,x\right)}{(x+1+y)(2\,y+2\,x+1)^2}$

18. $y' = \frac{(y+2\,x)^2}{x^2}$

19. $y' = \frac{(2\,y+x)^2}{y^2}$

20. $y' = \frac{x^2}{(y+4\,x)^2}$

Bernoulli's Equation. Identify the exponent n in Bernoulli's equation $y' + p(x)y = q(x)y^n$ and solve for $y(x)$.

21. $y^{-2}y' = 1 + x$

22. $yy' = 1 + x$

23. $y^{-2}y' + y^{-1} = 1 + x$

24. $yy' + y^2 = 1 + x$

25. $y' + y = y^{1/3}$

26. $y' + y = y^{1/5}$

27. $y' - y = y^{-1/2}$

28. $y' - y = y^{-1/3}$

29. $yy' + y^2 = e^x$

30. $y' + y = e^{2x}y^2$

Integrating Factor x^ay^b. Report an implicit solution for the given equation $Mdx + Ndy = 0$, using an integrating factor $Q = x^ay^b$. Follow Example **??**, page **??**.

31. $M = 3xy - 6y^2$, $N = 4x^2 - 15xy$

32. $M = 3xy - 10y^2$, $N = 4x^2 - 25xy$

33. $M = 2\,y - 12\,xy^2$, $N = 4\,x - 20\,x^2 y$

34. $M = 2\,y - 21\,xy^2$, $N = 4\,x - 35\,x^2 y$

35. $M = 3\,y - 32\,xy^2$, $N = 4\,x - 40\,x^2 y$

36. $M = 3\,y - 20\,xy^2$, $N = 4\,x - 25\,x^2 y$

37. $M = 12\,y - 30\,x^2 y^2$, $N = 12\,x - 25\,x^3 y$

38. $M = 12\,y + 90\,x^2 y^2$, $N = 12\,x + 75\,x^3 y$

39. $M = 15\,y + 90\,xy^2$, $N = 12\,x + 75\,x^2 y$

40. $M = 35\,y + 30\,xy^2$, $N = 28\,x + 25\,x^2 y$.

Integrating Factor e^{ax+by}. Report an implicit solution $U(x,y) = c$ for the given equation $M dx + N dy = 0$ using an integrating factor $Q = e^{ax+by}$. Follow Example **??**, page **??**.

41. $M = e^x + 2e^{2y}$, $N = e^x + 5e^{5y}$

42. $M = 3e^x + 2e^y$, $N = 4e^x + 5e^y$

43. $M = 12\,e^x + 2$, $N = 20\,e^x + 5$

44. $M = 12\,e^x + 2\,e^{-y}$, $N = 24\,e^x + 5\,e^{-y}$

45. $M = 12\,e^y + 2\,e^{-x}$, $N = 24\,e^y + 5\,e^{-x}$

46. $M = 12\,e^{-2\,y} + 2\,e^{-x}$, $N = 12\,e^{-2\,y} + 5\,e^{-x}$

47. $M = 16\,e^y + 2\,e^{-2\,x+3\,y}$, $N = 12\,e^y + 5\,e^{-2\,x+3\,y}$

48. $M = 16\,e^{-y} + 2\,e^{-2\,x-3\,y}$, $N = -12\,e^{-y} - 5\,e^{-2\,x-3\,y}$

49. $M = -16 - 2\,e^{2\,x+y}$, $N = 12 + 4\,e^{2\,x+y}$

50. $M = -16\,e^{-3\,y} - 2\,e^{2\,x}$, $N = 8\,e^{-3\,y} + 5\,e^{2\,x}$

Integrating Factor $Q(x)$. Report an implicit solution $U(x,y) = c$ for the given equation, using an integrating factor $Q = Q(x)$. Follow Example **??**, page **??**.

51. $(x + 2y)dx + (x - x^2)dy = 0$

52. $(x + 3y)dx + (x - x^2)dy = 0$

53. $(2x + y)dx + (x - x^2)dy = 0$

54. $(2x + y)dx + (x + x^2)dy = 0$

55. $(2x + y)dx + (2x + x^2)dy = 0$

56. $(x + y)dx + (2x + x^2)dy = 0$

57. $(x + y)dx + (3x + x^2)dy = 0$

58. $(x + y)dx + (3x + 5x^2)dy = 0$

59. $(x + y)dx + (3x)dy = 0$

60. $(x + y)dx + (7x)dy = 0$

Integrating Factor $Q(y)$.

61. $(y - y^2)dx + (x + y)dy = 0$

62. $(y - y^2)dx + (2x + y)dy = 0$

63. $(y - y^2)dx + (2x + 3y)dy = 0$

64. $(y + y^2)dx + (2x + 3y)dy = 0$

65. $(y + y^2)dx + (5x + 3y)dy = 0$

66. $(y + 5y^2)dx + (5x + 3y)dy = 0$

67. $(2y + 5y^2)dx + (5x + 3y)dy = 0$

68. $(2y + 5y^2)dx + (7x + 11y)dy = 0$

69. $(2y + 5y^3)dx + (3x + 7y)dy = 0$

70. $(3y + 5y^3)dx + (7x + 9y)dy = 0$