

# Phase Portraits for $\vec{u}' = A\vec{u}$

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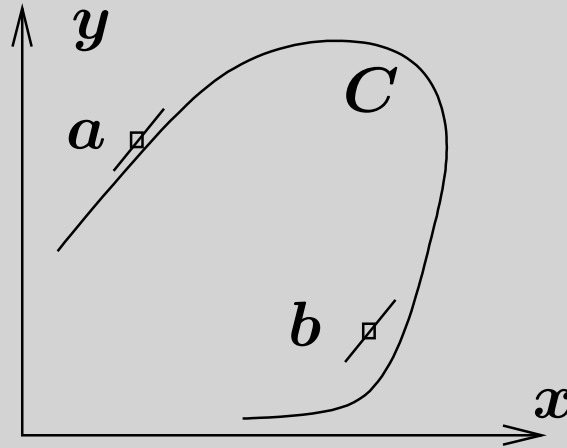
## How to Construct a Phase Portrait

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- Equilibria Plot in the  $xy$ -plane all equilibria of  $\vec{u}' = \vec{f}(\vec{u})$ .
- Window Select an  $x$ -range and a  $y$ -range for the graph window which includes all significant equilibria (Figure 3).
- Grid Plot a uniform grid of  $N$  grid points ( $N \approx 50$  for hand work) within the graph window, to populate the graphical white space (Figure 4). The isocline method might also be used to select grid points.
- Field Draw at each grid point a short tangent vector, a **replacement curve** for a solution curve through a grid point on a small time interval (Figure 5).
- Orbits Draw additional threaded trajectories on long time intervals into the remaining white space of the graphic (Figure 6). This is guesswork, based upon tangents to threaded trajectories matching nearby field tangents drawn in the previous step. See Figure 1 for matching details.

## Badly Threaded Solution Curves

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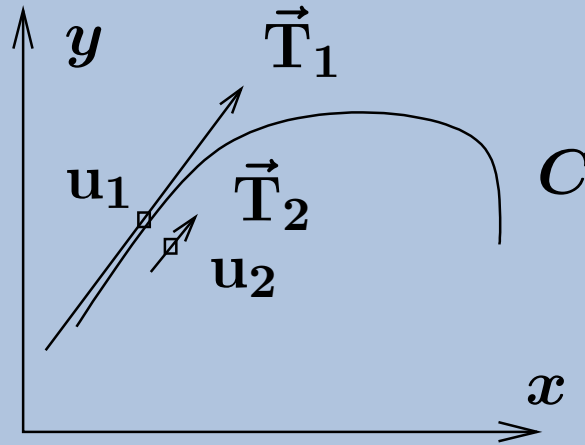


**Figure 1.** Badly threaded orbit.

Threaded solution curve  $C$  correctly matches its tangent to the tangent at nearby grid point  $a$ , but it fails to match at grid point  $b$ .

## Solution Curve Tangent Matching

Why does a threaded solution curve tangent  $\vec{T}_1$  have to match a tangent  $\vec{T}_2$  at a nearby grid point (see Figure 2)? A tangent vector is given by  $\vec{T} = \vec{u}'(t) = \vec{f}(\vec{u}(t))$ . Then  $\vec{T}_1 = \vec{f}(\vec{u}_1)$ ,  $\vec{T}_2 = \vec{f}(\vec{u}_2)$ . However,  $\vec{u}_1 \approx \vec{u}_2$  in the graphic, hence by continuity of  $\vec{f}$  it follows that  $\vec{T}_1 \approx \vec{T}_2$ .



**Figure 2.** Tangent matching.

Threaded solution curve  $C$  matches its tangent  $\vec{T}_1$  at  $u_1$  to direction field tangent  $\vec{T}_2$  at nearby grid point  $u_2$ .

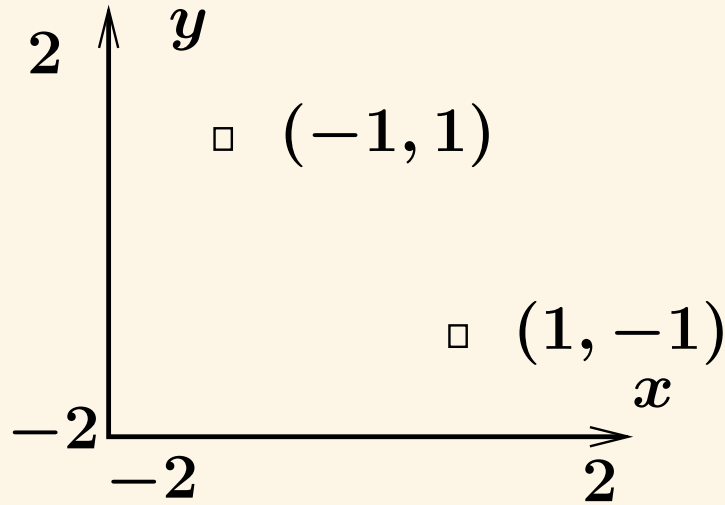
## Phase Portrait Illustration

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Consider the illustration

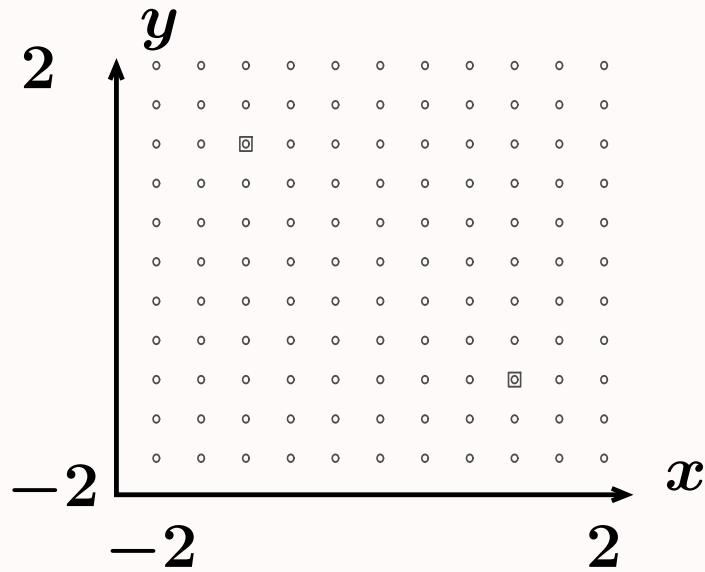
$$(1) \quad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

- The equilibria are  $(1, -1)$  and  $(-1, 1)$ .
- The graph window is selected as  $|x| \leq 2, |y| \leq 2$ , in order to include both equilibria.
- The uniform grid will be  $11 \times 11$ , although for hand work  $5 \times 5$  is normal.
- Tangents at the grid points are short line segments which do not touch each other – they are graphically the same as short solution curves.



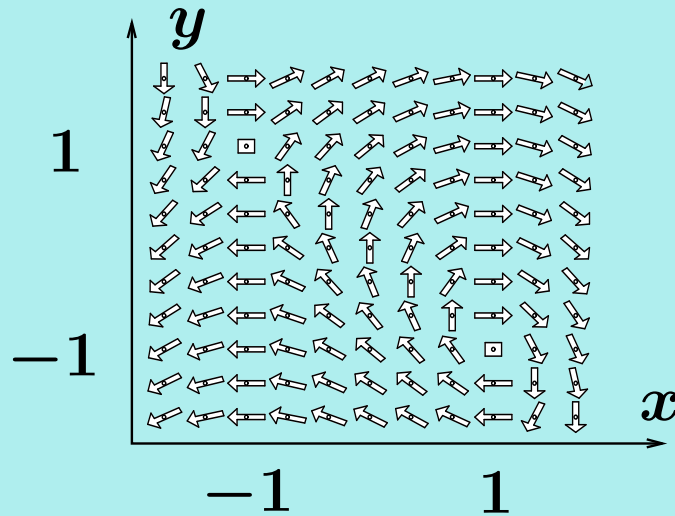
**Figure 3.** Equilibria  $(1, -1)$ ,  $(-1, 1)$  for (1) and graph window.

The equilibria  $(x, y)$  are calculated from equations  $0 = x + y$ ,  $0 = 1 - x^2$ . The graph window  $|x| \leq 2$ ,  $|y| \leq 2$  is invented initially, then updated until Figure 5 reveals sufficiently rich field details.



**Figure 4.** Equilibria  $(1, -1)$ ,  $(-1, 1)$  for (1) with  $11 \times 11$  uniform grid.

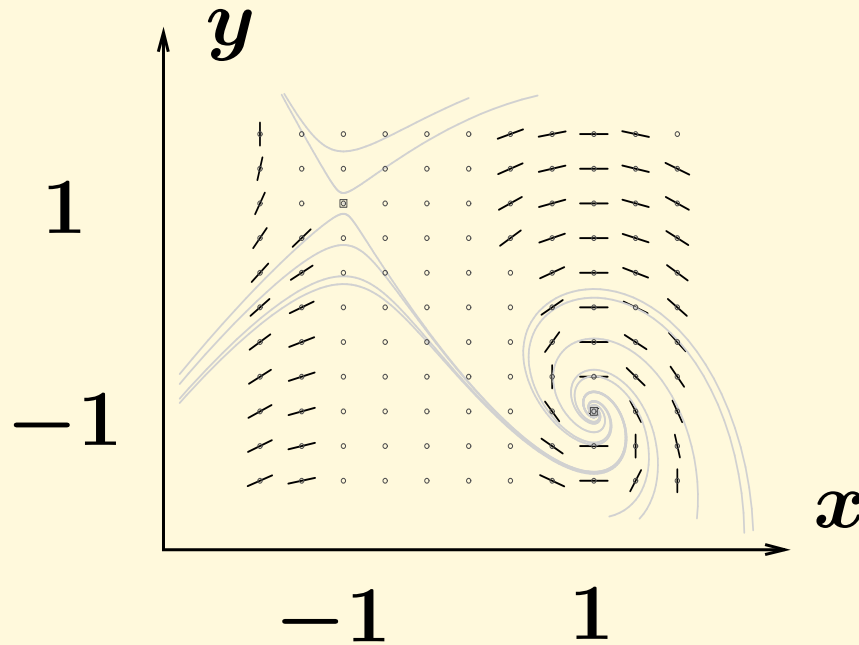
The equilibria (squares) happen to cover up two grid points (circles). The size  $11 \times 11$  is invented to fill the white space in the graphic.



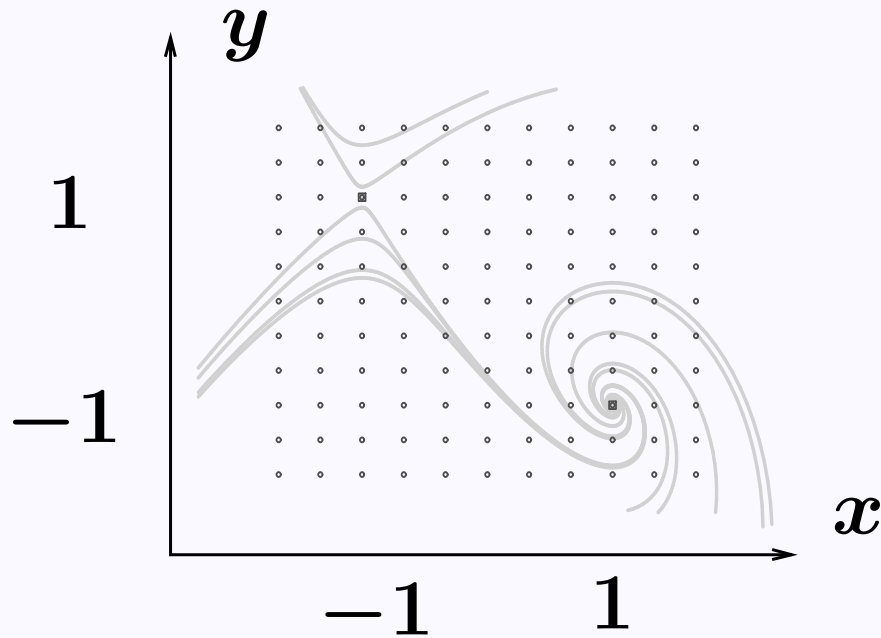
**Figure 5.** Equilibria  $(1, -1)$ ,  $(-1, 1)$  for (1).

Graph window  $|x| \leq 2$ ,  $|y| \leq 2$  with  $11 \times 11$  uniform grid and full direction field. An arrow shaft at a grid point represents a solution curve over a small time interval. Threaded solution curves on long time intervals have tangents nearly matching nearby arrow shaft directions.





**Figure 6.** Equilibria  $(1, -1)$ ,  $(-1, 1)$  for (1). Graph window with  $11 \times 11$  uniform grid and partial direction field. Threaded solution curves have been added. Arrow shafts selected from certain direction field arrows. Threaded solution curve tangents nearly match nearby direction field arrow shafts. See Figure 1 for how to match tangents.



**Figure 7.** Phase portrait for (1).

The direction field has been removed for clarity. Shown are threaded solution curves and an  $11 \times 11$  grid. Threaded solution curves do not actually cross, even though graphical resolution might suggest otherwise.

## Phase plot by computer

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Illustrated here is how to make a phase plot for the previous illustration using computer algebra system `maple`.

Before the computer work begins, the differential equation is defined and the equilibria are computed. Defaults supplied by `maple` allow an initial phase portrait to be plotted, from which the graph window is selected. The initial plot code:

```
with(DEtools):  
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:  
wind:=x=-2..2,y=-2..2:  
DEplot({des},[x(t),y(t)],t=-20..20,wind);
```

## Revised Computer Phase plot

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The initial plot suggests which initial conditions near the equilibria should be selected in order to create typical orbits on the graphic. The final code with initial data and options:

```
with(DEtools):
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:
opts:=stepsize=0.05,dirgrid=[13,13],
axes=none,thickness=3,arrows=small:
ics:=[[x(0)=-1,y(0)=1.1],[x(0)=-1,y(0)=1.5],
[x(0)=-1,y(0)=.9],[x(0)=-1,y(0)=.6],[x(0)=-1,y(0)=.3],
[x(0)=1,y(0)=-0.9],[x(0)=1,y(0)=-0.6],[x(0)=1,y(0)=-0.6],
[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=-1.6],[x(0)=1,y(0)=-1.3],
[x(0)=1,y(0)=-1.1]]:
DEplot({des},[x(t),y(t)],t=-20..20,wind,ics,opts);
```

## Interactive Maple Computer Plot

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An interactive phase portrait tool can be accessed in MAPLE 12 or 13 as follows:

```
?Task, PhasePortrait
```

```
Then: COPY TO CLIPBOARD
```

Then the copy is pasted into a worksheet, to begin the work.

Below is the example  $x' = x + y$ ,  $y' = 1 - x^2$ . Equilibria (1,-1), (-1,1) classify as a spiral and a saddle.

```
Plot Window: -2 < x < 2, -2 < y < 2
```

```
Differential Equations
```

```
x + y
```

```
1- x^2
```

```
Equilibrium Points: [1, -1], [-1, 1]
```

```
Parameter: -5 < t < 5
```

After entering data, hit **Enter Data** and then click on screen locations to add threaded solution curves.