Name *KEY ver 1*

Scores
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Differential Equations and Linear Algebra 2250

Midterm Exam 2

Version 1, 22 Mar 2012

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} c-a & -1 & 0 \\ a-c & 1 & a \\ 4a-2c & 3 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ b^2-b \\ b^2 \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution.

(b) [30%] Explain why $a=0$ and $b\neq 0$ implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why $a=b=0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

(a) $A\vec{x}=\vec{B}$ has a unique solution $\Leftrightarrow$ $|A|\neq 0$.

$$\begin{pmatrix} c-a & -1 & 0 & b \\ a-c & 1 & a & b^2-b \\ 4a-2c & 3 & a & b^2 \end{pmatrix} \xrightarrow[\text{combo}(1,3,3)]{\text{combo}(1,2,1)} \begin{pmatrix} c-a & -1 & 0 & b \\ 0 & 0 & a & b^2 \\ c+a & 0 & a & b^2+3b \end{pmatrix}$$

Then $|A| = (-1)(-1)\begin{vmatrix} 0 & a \\ c+a & a\end{vmatrix} = -a(c+a)$. $\boxed{\text{Unique sol} \Leftrightarrow a(c+a)\neq 0}$

(b) Then the last frame above is

$$\begin{pmatrix} c & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \\ c & 0 & 0 & b^2+3b \end{pmatrix}$$

The 2nd equation is $0=b^2$, a signal eq, meaning <u>No Sol for $a=0$ and $b\neq 0$</u>

(c) Then the last frame is

$$\begin{pmatrix} c & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$$

A free variable is z, because it does not appear in the system. Zero is always a sol, so the system is consistent. Therefore, it has ∞ - many solutions.

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2. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace S of $\mathcal{R}^3$ containing none of the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$.
$S = \text{span}(\vec{0})$ contains none of them.

(b) [10%] True or false: The set of solutions $\vec{x}$ in $\mathcal{R}^3$ of a consistent matrix equation $A\vec{x} = \vec{b}$ can equal all vectors in $\mathcal{R}^3$. Let A = zero matrix, $\vec{b}$ = zero vector.

(c) [10%] True or false Relations $xy = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$. See below

(d) [10%] True or false: Equations $x + y = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$. See below

(e) [20%] Two linear algebra theorems are able to conclude that the set S of all linear combinations of the functions $\sin(2x)$, e^x, $\cosh(2x)$ is a vector space of functions. State the the two theorems.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 10x_2 & + & 2x_3 & + & 4x_4 & = & 0, \\ 2x_1 & + & 4x_2 & + & x_3 & + & 2x_4 & = & 0, \\ -2x_1 & + & 4x_2 & & & & & = & 0, \\ 2x_1 & + & 12x_2 & + & 2x_3 & + & 4x_4 & = & 0. \end{array}$$

(c) False, by not a subspace Theorem: $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ satisfy the relations but their sum $= \begin{pmatrix}1\\1\\-1\end{pmatrix}$ fails to satisfy $xy = 0$.

(d) true, by the Kernel Theorem, because the relations are homogeneous linear algebraic equations of vector-matrix form $A\vec{x} = \vec{0}$.

(e) Theorem 1. The span of a set of vectors in a vector space V is a subspace S of V. (span Theorem)

Theorem 2. If S satisfies 1, 2, 3 below, then S is a subspace of vector space V. (subspace criterion)

1. $\vec{0}$ is in S; 2. $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x}+\vec{y}$ in S; 3. $\vec{x}$ in S, c = scalar $\Rightarrow c\vec{x}$ in S.

(f) The augmented matrix has rref

$$\left(\begin{array}{cccc|c} 1 & 0 & 1/4 & 1/2 & 0 \\ 0 & 1 & 1/8 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_1 + \frac{1}{4}x_3 + \frac{1}{2}x_4 = 0 \\ x_2 + \frac{1}{8}x_3 + \frac{1}{4}x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \to \vec{x} = \begin{pmatrix} -\frac{1}{4}x_3 - \frac{1}{2}x_4 \\ -\frac{1}{8}x_3 - \frac{1}{4}x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Then

$$\text{Subspace} = \text{Span}\left(\frac{\partial\vec{x}}{\partial x_3}, \frac{\partial\vec{x}}{\partial x_4}\right) = \text{Span}\left(\begin{pmatrix}-1/4\\-1/8\\1\\0\end{pmatrix}, \begin{pmatrix}-1/2\\-1/4\\0\\1\end{pmatrix}\right)$$

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$$\text{Basis} = \frac{\partial\vec{x}}{\partial x_3}, \frac{\partial\vec{x}}{\partial x_4} = \begin{pmatrix}-1/4\\-1/8\\1\\0\end{pmatrix}, \begin{pmatrix}-1/2\\-1/4\\0\\1\end{pmatrix}.$$

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3. **(Independence and Dependence)** Do **all parts**.

(a) [10%] State an independence test for 3 vectors in $\mathcal{R}^4$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State another [different than (a)] independence test for 3 vectors in $\mathcal{R}^4$.

(c) [10%] For any matrix A, **nullity**(A) equals the number of free variables for the problem $A\vec{x} = \vec{0}$. How many pivot columns in a 10×10 matrix A with **nullity**$(A) = 3$?

(d) [30%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & -6 & 0 \\ 0 & 2 & 0 & 5 & 1 \end{pmatrix}.$$

Decide if the four rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are independent and display the details of the chosen independence test.

(e) [40%] Extract from the list below a largest set of independent vectors.

$$\vec{v_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v_5} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \\ 9 \end{pmatrix}, \vec{v_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

(a) Three vectors in $\mathbb{R}^4$ are independent $\Leftrightarrow$ the augmented matrix A of the three vectors has rank $= 3$.

(b) • Solve $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ for c_1, c_2, c_3. The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent $\Leftrightarrow$ $c_1 = c_2 = c_3 = 0$.

• A determinant test: Let B be derived from A in part (a) by removing one row. If $|B| \neq 0$, then the vectors are independent. [However, this test is not $\Leftrightarrow$.]

(c) rank + nullity = # vars = 10, so rank = 7. The number of pivots = rank = 7.

(d) • Add rows 1, 2 to get row 4. The rows are dependent.

• Or, compute $|A| = 0 \Leftrightarrow$ dependence of the rows.

(e) Form the augmented matrix of $\vec{v}_1$ to $\vec{v}_6$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & -1 & -2 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 1 & 9 & 2 \end{pmatrix}. \text{ The } \mathrm{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & -\frac{3}{2} \\ & & \text{4 zero rows} & & & \end{pmatrix}.$$

pivots = 2, 3.

$\boxed{\text{Largest indep set} = \{\vec{v}_2, \vec{v}_3\}.}$

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