Linear Equation Method

$$y' + p(x)y = q(x)$$

- Write the DE in standard form $y' + p(x)y = q(x)$
  The method applies only if this is possible!

- Evaluate and simplify $\mathbb{P}(x) = \int p(x)dx$. Choose the constant of integration to simplify $e^{\mathbb{P}}$.

- Replace $y' + p(x)y$ by $\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$

- Clear fractions. Apply the method of quadrature.

Variation of parameters Formula

$$y' + p(x)y = q(x)$$

$y = y_h + y_p$ Superposition

Homogeneous solution. $y_h = c\,e^{-\mathbb{P}(x)}$ $c =$ constant

$\mathbb{P}(x) = \int p(x)dx$

particular solution. $y_p = e^{-\mathbb{P}(x)} \int q(r)e^{\mathbb{P}(r)}dr$

1.5: Example

Solve $\begin{cases} x' = t - \frac{2}{t}x \\ x(1) = 2 \end{cases}$

Ref: Edwards-Penney
See Section 1.5
and exercise 1.5-5, 1.5-10

$$x' + \frac{2}{t}x = t$$

std form is
$y' + p(x)y = q(x)$

$$\begin{aligned} Q &= \exp\left(\int \frac{2}{t}\,dt\right) \\ &= \exp(2\ln t) \\ &= \exp(\ln t^2) \\ &= t^2 \end{aligned}$$

Integrating factor is
$Q = \exp(\int p(x)dx)$

DE
replaced

$$\frac{(t^2x)'}{t^2} = t$$

Replace the LHS of the DE
$x' + \frac{2}{t}x = t$ by $\frac{(Qx)'}{Q}$

$$(t^2x)' = t^3$$

method of Quadrature applies
To This new replacement DE

Cross-multiply to clear fractions.

$$\int (t^2x)'dt = \int t^3 dt$$

Integrate both sides on t

$$t^2x = \frac{t^4}{4} + c$$

Fund. Thm of calculus applied to both sides.

$$x = \frac{t^2}{4} + ct^{-2}$$

Divide by coeff of $x(t)$

$$2 = \frac{1^2}{4} + c \cdot 1^{-2}$$
$$c = 7/4$$

Substitute $x=2, t=1$
solve for c

$$\boxed{x(t) = \frac{t^2}{4} + \frac{7}{4}t^{-2}}$$

Final answer
[Checked on scratch pad]

1.5: Example

Solve the linear problem

$$y' = y + e^x$$

Std. form

$y' + (-1)y = e^x$ — Form $y' + p(x)y = q(x)$

Factor $e^{\mathbb{P}}$

$\mathbb{P} = \int p(x)\,dx$

$= \int (-1)\,dx$

$= -x$ — Drop constant of integration

$e^{\mathbb{P}} = e^{-x}$ — simplified integrating factor

Quadrature form

$y' + (-1)y = e^x$ — std. form from above.

$\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = e^x$ — Replace LHS by $\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$.

$\dfrac{(e^{-x}y)'}{e^{-x}} = e^x$ — Replace $e^{\mathbb{P}}$ by e^{-x}.

$(e^{-x}y)' = 1$ — Cross-multiply, simplify to quadrature form.

Method of Quadrature

$\int (e^{-x}y)'\,dx = \int dx$ — Integrate both sides of the quadrature form.

$e^{-x}y = x + c$

$\boxed{y = ce^x + xe^x}$ — General solution $y = y_h + y_p$.

answer check next...

p53 # 3. $y' + 3y = 2x e^{-3x}$. Solve by the factorization method.

$$y' + (3)y = 2x e^{-3x}$$ standard form $y' + py = q$

$$\mathbb{P} = \int (3)\,dx$$ primitive $\mathbb{P} = \int p\,dx$

$$= 3x$$ simplify constants

$$e^{\mathbb{P}} = e^{3x}$$ simplified $e^{\mathbb{P}}$

Find Quadrature form

$$y' + (3)y = 2x e^{-3x}$$ std form

$$\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = 2x e^{-3x}$$ Replace $y' + py$ by $\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$

$$\frac{(e^{3x}y)'}{e^{3x}} = 2x e^{-3x}$$ substitute $e^{\mathbb{P}} = e^{3x}$

$$(e^{3x}y)' = 2x$$ use $e^{3x}e^{-3x} = e^0 = 1$ after cross-multiply.

Apply method of Quadature

$$\int (e^{3x}y)'\,dx = \int 2x\,dx$$ Apply quadrature to the quadrature form above

$$e^{3x}y = x^2 + c$$ Fund. Thm. of calculus

$$y = (x^2+c)e^{-3x}$$ Divide

Report ans and check

$$\boxed{y = (x^2+c)e^{-3x}}$$

ans checks with textbook

p 53 #5 $xy' + 2y = 3x$, $y(1) = 5$ Solve by the factorization method.

$y' + \left(\frac{2}{x}\right)y = 3$ — Standard form $y'+py=q$

$\mathbb{P} = \int \left(\frac{2}{x}\right)dx$ — primitive $\mathbb{P} = \int p\,dx$

$= 2\ln x$

$= \ln x^2$

$e^{\mathbb{P}} = e^{\ln x^2}$

$= x^2$ — simplified $e^{\mathbb{P}}$ found

Find Quadrature Form

$y' + \left(\frac{2}{x}\right)y = 3$ — std form copied

$\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = 3$ — Replace $y'+py$ by $\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$

$\frac{(x^2y)'}{x^2} = 3$ — Substitute x^2 for $e^{\mathbb{P}}$

$(x^2y)' = 3x^2$ — Quadrature form found

Method of Quadrature

$\int (x^2y)'dx = \int 3x^2 dx$ — Apply quadrature to the previous line.

$x^2y = x^3 + c$

$y = x + c/x^2$ — Divide. Solution candidate found.

Report answer and check

$5 = 1 + \frac{c}{1^2}$ — Substitute $x=1, y=5$ to find $c=4$.

answer $\boxed{y = x + \frac{4}{x^2}}$

answer checks with text.

P53 # 11 $xy' + y = 3xy$, $y(1) = 0$ Solve for $y(x)$ by the factorization method.

$y' + (\frac{1}{x})y = 3y$ Divide by x

$y' + (\frac{1}{x} - 3)y = 0$ Standard form $y' + py = q$

$\mathbb{P} = \int (\frac{1}{x} - 3)dx$ primitive $\mathbb{P} = \int p\,dx$

$= \ln x - 3x$

$e^{\mathbb{P}} = e^{\ln x} e^{-3x}$

$= x e^{-3x}$ simplified $e^{\mathbb{P}}$

Find Quadrature Form

$y' + (\frac{1}{x} - 3)y = 0$ Copy of std form

$\frac{(e^{\mathbb{P}} y)'}{e^{\mathbb{P}}} = 0$ Replace $y' + py$ by $\frac{(e^{\mathbb{P}} y)'}{e^{\mathbb{P}}}$

$(e^{\mathbb{P}} y)' = 0$ cross-multiply

$(x e^{-3x} y)' = 0$ Substitute for $e^{\mathbb{P}}$. Quadrature form found.

method of Quadrature

$\int (x e^{-3x} y)' dx = \int 0\, dx$ method of quadrature applied

$x e^{-3x} y = c$

$y = \frac{c}{x} e^{3x}$ Divide. Candidate Solution found.

Report answer and check

$0 = \frac{c}{1} e^{3}$ Substitute $x = 1$, $y = 0$ [from $y(1) = 0$] to find $c = 0$.

$\boxed{y = 0}$

ans checks with book.

Brine Mixing

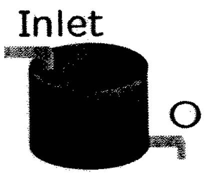


Figure 1. A brine tank with one inlet and one outlet.

A given tank contains brine, that is, water and salt. Input pipes supply other, possibly different brine mixtures at varying rates, while output pipes drain the tank. The problem is to determine the salt $x(t)$ in the tank at any time. The basic chemical law to be applied is the **mixture law**

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

The law is applied under a simplifying assumption: *the concentration of salt in the brine is uniform throughout the fluid.* Stirring is one way to meet this requirement.

One Input and One Output. Let the input be $a(t)$ liters per minute with concentration C_1 kilograms of salt per liter. Let the output empty $b(t)$ liters per minute. The tank is assumed to contain V_0 liters of brine at $t = 0$. The tank gains fluid at rate $a(t)$ and loses fluid at rate $b(t)$, therefore $V(t) = V_0 + \int_0^t [a(r) - b(r)]dr$ is the volume of brine in the tank at time t. The *mixture law* applies to obtain the model linear differential equation

$$\frac{dx}{dt} = C_1 a(t) - \frac{b(t)x(t)}{V(t)}.$$

p 53 #33. A tank contains 1000 liters of brine, 100 kg is salt. Pure water enters at 5 liters/second. Uniformly mixed brine exits at 5 liters/second. Find the time t when the amount $x(t)$ of salt equals 10 kg.

Model

Apply $\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x$ where $r_o = r_i = 5$, $c_i = 0$, $V = 1000$ and $x(0) = 100$. Then

$$\begin{cases} \frac{dx}{dt} = 0 - \frac{5}{1000} x \\ x(0) = 100 \end{cases}$$

is the model.

Solve the model.

This is a growth-decay model. The solution is

$$x(t) = x(0) e^{-5t/1000}$$
$$= 100 e^{-t/200}$$

Find time t.

$10 = 100 e^{-t/200}$ Need $x(t) = 10$

$0.1 = e^{-t/200}$

$\ln 0.1 = -t/200$

$t = 200 \ln 10$ $0.1 = \frac{1}{10}$ and $\ln \frac{1}{10} = -\ln 10$.

$= 461$ seconds

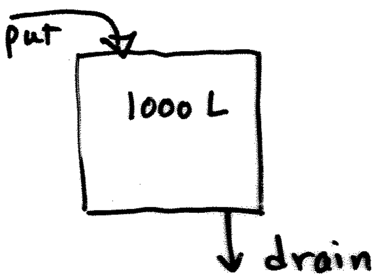
Report answer and check

461 seconds

Ans checks with book (7 min 41 sec)

33

A tank contains 1000 Liters of solution, with 100 kilograms of dissolved salt. Pure water pumps into the Tank at rate 5 Liters / second. It exits the Tank at 5 liters/second. Assume uniform mixing. How long before only 10 kilograms of salt remains in the Tank?



$x(t)$ = amount of salt in the Tank at time t

Model $\begin{cases} x' = -\left(\frac{1}{200}\right)x \\ x(0) = 100 \end{cases}$

Solve model By Growth - Decay recipe $x(t) = 100 e^{-t/200}$

Solve $x(t) = 10$ for t

$$x(t) = 10$$
$$100\, e^{-t/200} = 10$$
$$e^{-t/200} = 0.1$$
$$-t/200 = \ln 0.1$$
$$t = 200(-1)\ln\frac{1}{10}$$
$$= 200 \ln\left(\frac{1}{10}\right)^{-1}$$
$$= 200 \ln 10$$
$$= 461 \text{ seconds}$$
$$= 7 \text{ min } + 41 \text{ sec}$$