Vector Spaces and Subspaces

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Vector Space V

It is a **data set** V plus a **toolkit** of eight (8) algebraic properties. The data set consists of packages of data items, called **vectors**, denoted $\vec{X}$, $\vec{Y}$ below.

Closure	The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$	commutative
	$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$	negative
Scalar multiply	$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$	distributive I
	$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X}) = (k_1k_2)\vec{X}$	distributive III
	$1\vec{X} = \vec{X}$	identity

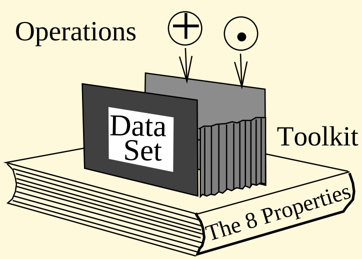

Figure 1. A *Vector Space* is a data set, operations $\boxplus$ and $\boxdot$, and the 8-property toolkit.

Definition of Subspace

A **subspace** S of a vector space V is a nonvoid subset of V which under the operations $+$ and $\cdot$ of V forms a vector space in its own right.

Subspace Criterion

Let S be a subset of V such that

1. Vector $\vec{0}$ is in S.
2. If $\vec{X}$ and $\vec{Y}$ are in S, then $\vec{X} + \vec{Y}$ is in S.
3. If $\vec{X}$ is in S, then $c\vec{X}$ is in S.

Then S is a subspace of V.

Items **2**, **3** can be summarized as *all linear combinations of vectors in S are again in S*. In proofs using the criterion, items 2 and 3 may be replaced by

$$c_1\vec{X} + c_2\vec{Y} \text{ is in } S.$$

Subspaces are Working Sets

We call a subspace S of a vector space V a **working set**, because the purpose of identifying a subspace is to shrink the original data set V into a smaller data set S, customized for the application under study.

A Key Example. Let $V = R^3$ and let S be the plane of action of a planar kinematics experiment, a slot car on a track. The data set D for the experiment is all 3-vectors $\vec{v}$ in V collected by a data recorder. Detected by GPS and recorded by computer is the 3D position of the slot car, which would be planar, except for bumps in the track. The original data set D will be transformed into a new data set D_1 that lies entirely in a plane. Plane geometry computations then proceed with the Toolkit for V, on the smaller planar data set D_1.
How to create D_1? The ideal plane of action S is computed as a homogeneous equation (like 2x+3y+1000z=0), the equation of a plane. Least squares is applied to data set D to find an optimal equation for S. Altered data set D_1 is created from D (and D discarded) to artificially satisfy the plane equation. Then D_1 is contained in S (D obviously was not). The smaller storage set S replaces the larger storage set V. The customized smaller set S is the *working set* for the kinematics problem.

The Kernel Theorem

Theorem 1 (Kernel Theorem)
Let $\boldsymbol{V}$ be one of the vector spaces $\boldsymbol{R}^n$ and let $\boldsymbol{A}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix. Define a smaller set $\boldsymbol{S}$ of data items in $\boldsymbol{V}$ by the kernel equation

$$\boldsymbol{S} = \{\vec{x} : \vec{x} \text{ in } \boldsymbol{V}, \quad \boldsymbol{A}\vec{x} = \vec{0}\}.$$

Then $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$.
In particular, operations of addition and scalar multiplication applied to data items in $\boldsymbol{S}$ give answers back in $\boldsymbol{S}$, and the 8-property toolkit applies to data items in $\boldsymbol{S}$.

Proof: Zero is in $\boldsymbol{V}$ because $\boldsymbol{A}\vec{0} = \vec{0}$ for any matrix $\boldsymbol{A}$. To apply the subspace criterion, we verify that $\vec{z} = c_1\vec{x} + c_2\vec{y}$ for $\vec{x}$ and $\vec{y}$ in $\boldsymbol{V}$ also belongs to $\boldsymbol{V}$. The details:

$$\begin{aligned} A\vec{z} &= A(c_1\vec{x} + c_2\vec{y}) \\ &= A(c_1\vec{x}) + A(c_2\vec{y}) \\ &= c_1 A\vec{x} + c_2 A\vec{y} \\ &= c_1\vec{0} + c_2\vec{0} && \text{Because } A\vec{x} = A\vec{y} = \vec{0}, \text{ due to } \vec{x}, \vec{y} \text{ in } V. \\ &= \vec{0} && \text{Therefore, } A\vec{z} = \vec{0}, \text{ and } \vec{z} \text{ is in } V. \end{aligned}$$

The proof is complete.

Not a Subspace Theorem

Theorem 2 (Testing S not a Subspace)
Let V be an abstract vector space and assume S is a subset of V. Then S is not a subspace of V provided one of the following holds.

(1) The vector $\vec{0}$ is not in S.

(2) Some $\vec{x}$ and $-\vec{x}$ are not both in S.

(3) Vector $\vec{x} + \vec{y}$ is not in S for some $\vec{x}$ and $\vec{y}$ in S.

Proof: The theorem is justified from the *Subspace Criterion*.
1. The criterion requires $\vec{0}$ is in S.
2. The criterion demands $c\vec{x}$ is in S for all scalars c and all vectors $\vec{x}$ in S.
3. According to the subspace criterion, the sum of two vectors in S must be in S.

Definition of Independence and Dependence

A list of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a vector space V are said to be **independent** provided every linear combination of these vectors is uniquely represented. **Dependent** means **not independent**.

Unique representation

An equation

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$$

implies matching coefficients: $a_1 = b_1, \ldots, a_k = b_k$.

Independence Test

Form the system in unknowns $c_1, \ldots, c_k$

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Solve for the unknowns [how to do this depends on V]. Then the vectors are independent if and only if the unique solution is $c_1 = c_2 = \cdots = c_k = 0$.

Independence test for two vectors $\vec{v}_1, \vec{v}_2$

In an abstract vector space V, form the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}.$$

Solve this equation for the two constants c_1, c_2.

Then $\vec{v}_1, \vec{v}_2$ are independent in V if and only if the system has unique solution $c_1 = c_2 = 0$.

There is no algorithm for how to do this, because it depends on the vector space V and sometimes on detailed information obtained from bursting the data packages $\vec{v}_1, \vec{v}_2$. If V is some R^n, then combo-swap-mult sequences apply.

Geometry and Independence

- One fixed vector is independent if and only if it is nonzero.
- Two fixed vectors are independent if and only if they form the edges of a parallelogram of positive area.
- Three fixed vectors are independent if and only if they are the edges of a parallelepiped of positive volume.

In an abstract vector space $\boldsymbol{V}$, one vector [one data package] is independent if and only if it is a nonzero vector. Two vectors [two data packages] are independent if and only if one is not a scalar multiple of the other. There is no simple test for three vectors.

Illustration

Vectors $\vec{v}_1 = \cos x$ and $\vec{v}_2 = \sin x$ are two data packages [graphs] in the vector space $\boldsymbol{V}$ of continuous functions. They are independent because one graph is not a scalar multiple of the other graph.

An Illustration of the Independence Test

Two column vectors are tested for independence by forming the system of equations $\boldsymbol{c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}}$, e.g,

$$\boldsymbol{c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}.$$

This is a homogeneous system $\boldsymbol{A\vec{c} = \vec{0}}$ with

$$\boldsymbol{A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}.$$

The system $\boldsymbol{A\vec{c} = \vec{0}}$ can be solved for $\vec{c}$ by combo-swap-mult methods. Because $\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{I}$, then $\boldsymbol{c_1 = c_2 = 0}$, which verifies independence.
If the system $\boldsymbol{A\vec{c} = \vec{0}}$ is square, then $\mathbf{det}(\boldsymbol{A}) \neq \mathbf{0}$ applies to test independence.
Determinants cannot be used directly when the system is not square, e.g., consider the homogeneous system

$$\boldsymbol{c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}.$$

It has vector-matrix form $\boldsymbol{A\vec{c} = \vec{0}}$ with $\mathbf{3 \times 2}$ matrix $\boldsymbol{A}$, for which $\mathbf{det}(\boldsymbol{A})$ is undefined.

Rank Test

In the vector space $\boldsymbol{R}^n$, the independence test leads to a system of $\boldsymbol{n}$ linear homogeneous equations in $\boldsymbol{k}$ variables $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$. The test requires solving a matrix equation $\boldsymbol{A}\vec{\boldsymbol{c}} = \vec{\mathbf{0}}$. The signal for independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. To justify the various statements, we use the relation $\mathbf{nullity}(\boldsymbol{A}) + \mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$, where $\boldsymbol{k}$ is the column dimension of $\boldsymbol{A}$.

Theorem 3 (Rank-Nullity Test)
Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ be $\boldsymbol{k}$ column vectors in $\boldsymbol{R}^n$ and let $\boldsymbol{A}$ be the augmented matrix of these vectors. The vectors are independent if $\mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$ and dependent if $\mathbf{rank}(\boldsymbol{A}) < \boldsymbol{k}$. The conditions are equivalent to $\mathbf{nullity}(\boldsymbol{A}) = \mathbf{0}$ and $\mathbf{nullity}(\boldsymbol{A}) > \mathbf{0}$, respectively.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $A\vec{c} = \vec{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the adjugate formula $A^{-1} = \textbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 4 (Determinant Test)
Let A be a square augmented matrix of column vectors. The column vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

Sampling Test

Let functions $f_1, \ldots, f_n$ be given and let $x_1, \ldots, x_n$ be distinct x-sample values. Define

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$. Choose $x = x_1$ and $x = x_2$ in this relation to get $A\vec{c} = \vec{0}$, where $\vec{c}$ has components c_1, c_2. If $\det(A) \neq 0$, then A^{-1} exists, and this in turn implies $\vec{c} = A^{-1}A\vec{c} = \vec{0}$. We conclude f_1, f_2 are independent.

Wronskian Test

Let functions $f_1, \ldots, f_n$ be given and let x_0 be a given point. Define

$$W = \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}.$$

Then $\det(W) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions. The matrix W is called the **Wronskian matrix** of $f_1, \ldots, f_n$ and $\det(W)$ is called the **Wronskian determinant**.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$ and $c_1 f_1'(x) + c_2 f_2'(x) = 0$. Choose $x = x_0$ in this relation to get $W\vec{c} = \vec{0}$, where $\vec{c}$ has components c_1, c_2. If $\det(W) \neq 0$, then W^{-1} exists, and this in turn implies $\vec{c} = W^{-1}W\vec{c} = \vec{0}$. We conclude f_1, f_2 are independent.

Atoms

Definition. A **base atom** is one of the functions

$$1, \quad e^{ax}, \quad \cos bx, \quad \sin bx, \quad e^{ax}\cos bx, \quad e^{ax}\sin bx$$

where $b > 0$. Define an **atom** for integers $n \geq 0$ by the formula

$$\textbf{atom} = x^n(\textbf{base atom}).$$

The powers $1, x, x^2, \ldots$ are atoms (multiply base atom 1 by x^n). Multiples of these powers by $\cos bx$, $\sin bx$ are also atoms. Finally, multiplying all these atoms by e^{ax} expands and completes the list of atoms.

Alternatively, an **atom** is a function with coefficient 1 obtained as the real or imaginary part of the complex expression

$$x^n e^{ax}(\cos bx + i \sin bx).$$

Illustration

We show the powers $\mathbf{1}$, $\boldsymbol{x}$, $\boldsymbol{x^2}$, $\boldsymbol{x^3}$ are independent atoms by applying the Wronskian Test:

$$\boldsymbol{W} = \begin{pmatrix} \mathbf{1} & \boldsymbol{x_0} & \boldsymbol{x_0^2} & \boldsymbol{x_0^3} \\ \mathbf{0} & \mathbf{1} & \mathbf{2}\boldsymbol{x_0} & \mathbf{3}\boldsymbol{x_0^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{6}\boldsymbol{x_0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{6} \end{pmatrix}.$$

Then $\mathbf{det}(\boldsymbol{W}) = \mathbf{12} \neq \mathbf{0}$ implies the functions $\mathbf{1}$, $\boldsymbol{x}$, $\boldsymbol{x^2}$, $\boldsymbol{x^3}$ are linearly independent.

Subsets of Independent Sets are Independent

Suppose $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ make an independent set and consider the subset $\vec{v}_1$, $\vec{v}_2$. If

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

then also

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

where $c_3 = 0$. Independence of the larger set implies $c_1 = c_2 = c_3 = 0$, in particular, $c_1 = c_2 = 0$, and then $\vec{v}_1$, $\vec{v}_2$ are indpendent.

Theorem 5 (Subsets and Independence)

- A non-void subset of an independent set is also independent.
- Non-void subsets of dependent sets can be independent or dependent.

Atoms and Independence

Theorem 6 (Independence of Atoms)
Any list of distinct atoms is linearly independent.

Unique Representation

The theorem is used to extract equations from relations involving atoms. For instance:

$$(c_1 - c_2)\cos x + (c_1 + c_3)\sin x + c_1 + c_2 = 2\cos x + 5$$

implies

$$\begin{aligned} c_1 - c_2 &= 2, \\ c_1 + c_3 &= 0, \\ c_1 + c_2 &= 5. \end{aligned}$$

Atoms and Differential Equations

It is known that solutions of linear constant coefficient differential equations of order n and also systems of linear differential equations with constant coefficients have a general solution which is a linear combination of atoms.

- The harmonic oscillator $y'' + b^2y = 0$ has general solution $y(x) = c_1 \cos bx + c_2 \sin bx$. This is a linear combination of the two atoms $\cos bx$, $\sin bx$.
- The third order equation $y''' + y' = 0$ has general solution $y(x) = c_1 \cos x + c_2 \sin x + c_3$. The solution is a linear combination of the independent atoms $\cos x$, $\sin x$, 1.
- The linear dynamical system $x'(t) = y(t)$, $y'(t) = -x(t)$ has general solution $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = -c_1 \sin t + c_2 \cos t$, each of which is a linear combination of the independent atoms $\cos t$, $\sin t$.