Undetermined Coefficients, Resonance, Applications

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An Undetermined Coefficients Illustration

The differential equation $y'' - y = x + xe^x$ will be solved. Verified for general solution $y = y_h + y_p$ are the formulas

$$y_h = c_1 e^x + c_2 e^{-x}, \quad y_p = -x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.$$

Homogeneous solution. The homogeneous equation $y'' - y = 0$ has characteristic equation $r^2 - 1 = 0$, roots $r = \pm 1$, and atom list e^x, e^{-x}. Then $y_h = c_1 e^x + c_2 e^{-x}$.

Rule I trial solution. Let $f(x) = x + xe^x$. The derivatives f, f', f'', …have atom list

$$1, \quad x, \quad e^x, \quad xe^x.$$

Then $k = 4$ is the number of atoms in the trial solution. Because atom e^x is a solution of the homogeneous equation $y'' - y = 0$, then Rule I **FAILS**.

Rule II trial solution. Break up the atom list for f into groups with the same base atom, as follows.

Group	**Atoms**	**Base atom**
1	$1, x$	e^{0x}
2	e^x, xe^x	e^x

Group 1 is unchanged, because the first atom 1 is not a solution of the homogeneous equation $y'' - y = 0$. Group 2 has a FAIL, because the first atom e^x is a solution of the homogeneous equation $y'' - y = 0$, as seen in Rule I. We multiply group 2 by factor x, then the first atom is xe^x, which is not a solution of the homogeneous equation [Eulers theorem says xe^x is a solution if and only if 1 is a double root of the characteristic equation $r^2 - 1 = 0$; it isn't].

Group	**Atoms**
1	$1, x$
New 2	xe^x, x^2e^x

The trial solution, according to Rule II, is a linear combination of the atoms in the last table.

$$y = (d_1 + d_2 x) + (d_3 x e^x + d_4 x^2 e^x).$$

Substitute the trial solution into the DE

Substitute $y(d_1 + d_2 x) + (d_3 x e^x + d_4 x^2 e^x)$ **into** $y'' - y = x + xe^x$. The details:

$\text{LHS} = y'' - y$	Left side of the equation.
$= [y_1'' - y_1] + [y_2'' - y_2]$	Let $y = y_1 + y_2$, $y_1 = d_1 + d_2 x$, $y_2 = d_3 x e^x + d_4 x^2 e^x$.
$= [0 - y_1] + [2d_3 e^x + 2d_4 e^x + 4d_4 x e^x]$	Use $y_1'' = 0$ and $y_2'' = y_2 + 2d_3 e^x + 2d_4 e^x + 4d_4 x e^x$.
$= (-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x$	Collect on distinct atoms.

Write out a 4×4 system. Because LHS $=$ RHS and RHS $= \boldsymbol{x} + \boldsymbol{xe^x}$, the last display gives the relation

$$\begin{aligned} &(-d_1)1 + (-d_2)x + \\ &(2d_3 + 2d_4)e^x + (4d_4)xe^x = (0)1 + (1)x + (0)e^x + (1)xe^x. \end{aligned} \tag{1}$$

Equate coefficients of matching atoms left and right to give the system of equations

$$\begin{array}{rrrrcl} -d_1 & & & & = & 0, \\ & -d_2 & & & = & 1, \\ & & 2d_3 & +2d_4 & = & 0, \\ & & & 4d_4 & = & 1. \end{array} \tag{2}$$

Atom matching effectively removes x and changes the equation into a 4×4 linear system for symbols d_1, d_2, d_3, d_4.

Atom Matching Explained. The technique is independence. To explain, independence of atoms means that a linear combination of atoms is uniquely represented, hence two such equal representations must have matching coefficients. Relation (1) says that two linear combinations of the same list of atoms are equal. Hence coefficients left and right in (1) must match, which gives 4×4 system (2).

Solve the equations. The 4×4 system must always have a unique solution. Equivalently, there are four lead variables and zero free variables. Solving by back-substitution gives $d_1 = 0, d_2 = -1, d_3 = 1/4, d_4 = -1/4$.

Report y_p. The trial solution with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Report $y = y_h + y_p$

From above,

$$y_h = c_1 e^x + c_2 e^{-x}, \quad y_p = -x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.$$

Then $y = y_h + y_p$ is given by

$$y = c_1 e^x + c_2 e^{-x} - x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.$$

Answer check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```

Phase-amplitude conversion–I

Given a simple harmonic motion $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$, as in Figure 1, define **amplitude** A and **phase angle** α by the formulas

$$A = \sqrt{c_1^2 + c_2^2},\ c_1 = A\cos\alpha,\ c_2 = A\sin\alpha.$$

Then the simple harmonic motion has the **phase-amplitude form**

$$x(t) = A\cos(\omega t - \alpha). \tag{3}$$

To directly obtain (3) from trigonometry, use the trigonometric identity

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

with $a = \omega t$ and $b = \alpha$. It is known from trigonometry that $x(t)$ has **period** $2\pi/\omega$ and **phase shift** α/ω. A full period is called a **cycle** and a half-period a **semicycle**. The **frequency** $\omega/(2\pi)$ is the number of complete cycles per second, or the reciprocal of the period.

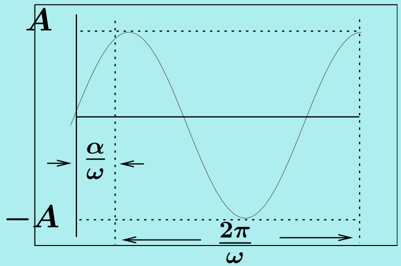

Figure 1. Simple harmonic oscillation $x(t) = A\cos(\omega t - \alpha)$, showing the period $2\pi/\omega$, the phase shift α/ω and the amplitude A.

Phase-amplitude conversion–II

- The **phase shift** is the amount of horizontal translation required to shift the cosine curve $\cos(\omega t - \alpha)$ so that its graph is atop $\cos(\omega t)$. To find the phase shift from $x(t)$, set the argument of the cosine term to zero, then solve for t.

- To solve for $\alpha \geq 0$ in the equations $c_1 = A\cos\alpha$, $c_2 = A\sin\alpha$, first compute numerically by calculator the radian angle $\phi = \arctan(c_2/c_1)$, which is in the range $-\pi/2$ to $\pi/2$. Quadrantal angle rules must be applied when $c_1 = 0$, because calculators return an error code for division by zero. A *common error* is to set α equal to ϕ. Not just the violation of $\alpha \geq 0$ results — the error is a fundamental one, due to trigonometric intricacies, causing us to consider the equations $c_1 = A\cos\alpha$, $c_2 = A\sin\alpha$ in order to construct the answer for α:

$$\alpha = \begin{cases} \phi & (c_1, c_2) \text{ in quadrant } I, \\ \phi + \pi & (c_1, c_2) \text{ in quadrant } II, \\ \phi + \pi & (c_1, c_2) \text{ in quadrant } III, \\ \phi + 2\pi & (c_1, c_2) \text{ in quadrant } IV. \end{cases}$$

Cafe door

Restaurant waiters and waitresses are accustomed to the cafe door, which partially blocks the view of onlookers, but allows rapid trips to the kitchen – see Figure 2. The door is equipped with a spring which tries to restore the door to the equilibrium position $\boldsymbol{x = 0}$, which is the plane of the door frame. There is a dampener attached, to keep the number of oscillations low.

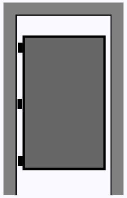

Figure 2. A cafe door on three hinges with dampener in the lower hinge. The equilibrium position is the plane of the door frame.

The top view of the door, Figure 3, shows how the angle $\boldsymbol{x(t)}$ from equilibrium $\boldsymbol{x = 0}$ is measured from different door positions.

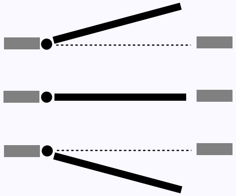

Figure 3. Top view of a cafe door, showing the three possible door positions.

Pet door

Designed for dogs and cats, the small door in Figure 4 allows animals to enter and exit the house freely. A pet door might have a weather seal and a security lock.

Figure 4. A pet door.

The equilibrium position is the plane of the door frame.

The pet door swings freely from hinges along the top edge. One hinge is spring–loaded with dampener. Like the cafe door, the spring restores the door to the equilibrium position while the dampener acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–dampener system is removed, then the door continues to oscillate! The cafe door model will not describe the pet door.

Cafe Door Model

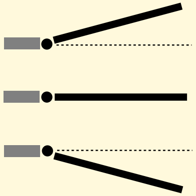

Figure 5. Top view of a cafe door, showing the three possible door positions.

Figure 5 shows that, for modeling purposes, the cafe door can be reduced to a torsional pendulum with viscous damping. This results in the **cafe door** equation

$$Ix''(t) + cx'(t) + \kappa x(t) = 0. \tag{4}$$

The removal of the spring ($\kappa = 0$) causes the solution $x(t)$ to be monotonic, which is a reasonable fit to a springless cafe door.

Pet Door Model

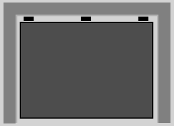

Figure 6. A pet door.
The equilibrium position is the plane of the door frame.

For modeling purposes, the pet door can be compressed to a linearized swinging rod of length L (the door height). The torque $I = mL^2/3$ of the door assembly becomes important, as well as the linear restoring force kx of the spring and the viscous damping force cx' of the dampener. All considered, a suitable model is the **pet door** equation

$$I\,x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right)x(t) = 0. \tag{5}$$

Derivation of (5) is by equating to zero the algebraic sum of the forces. Removing the dampener and spring ($c = k = 0$) gives a harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with $\omega^2 = 0.5mgL/I$, which establishes sanity for the modeling effort. Equation (5) is *formally* the cafe door equation with an added linearization term $0.5mgLx(t)$ obtained from $0.5mgL\sin x(t)$.

Classifying Damped Models

Consider a differential equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b, c. It has characteristic equation $ar^2 + br + c = 0$ with roots r_1, r_2.

Classification	Defining properties
Overdamped	Distinct real roots $r_1 \neq r_2$ Positive discriminant $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $=$ exponential $\times$ monotonic function
Critically damped	Double real root $r_1 = r_2$ Zero discriminant $x = c_1 e^{r_1 t} + c_2\, t\, e^{r_1 t}$ $=$ exponential $\times$ monotonic function
Underdamped	Complex conjugate roots $\alpha \pm i\,\beta$ Negative discriminant $x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$ $=$ exponential $\times$ harmonic oscillation

Pure Resonance

Graphed in Figure 7 are the envelope curves $x = \pm t$ and the solution $x(t) = t\sin 4t$ of the equation $x''(t) + 16x(t) = 8\cos\omega t$, where $\omega = 4$.

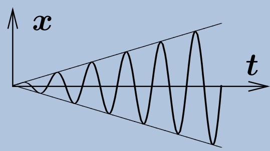

Figure 7. Pure resonance.

The notion of **pure resonance** in the differential equation

$$x''(t) + \omega_0^2\, x(t) = F_0 \cos(\omega t) \tag{6}$$

is the existence of a solution that is unbounded as $t \to \infty$. We already know that for $\omega \neq \omega_0$, the general solution of (6) is the sum of two harmonic oscillations, hence it is bounded. Equation (6) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution $x(t) = \dfrac{F_0}{2\omega_0}\, t\, \sin(\omega_0\, t)$. To summarize:

> Pure resonance occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω, in which case all solutions of the differential equation are unbounded.

In Figure 7, this is illustrated for $x''(t) + 16x(t) = 8\cos 4t$, which in (6) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$.

Real-World Damping Effects

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching

$$\omega = \omega_0 \equiv \sqrt{k/m}$$

characterizes the event. This ideal situation never happens in the physical world, because *damping is always present*. In the presence of damping $c > 0$, it can be established that *only bounded solutions exist* for the forced spring-mass system

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \tag{7}$$

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible solution amplitude. It can be shown that this happens for the condition

$$\omega = \sqrt{k/m - c^2/(2m^2)}, \quad k/m - c^2/(2m^2) > 0. \tag{8}$$

Pure resonance $\omega = \omega_0 \equiv \sqrt{k/m}$ is the limiting case obtained by setting the damping constant c to zero in condition (8). This strange but predictable interaction exists between the damping constant c and the size of solutions, relative to the external frequency ω, even though all solutions remain bounded.