## Fundamental Theorem of Calculus

Isaac Newton found these formulas in an effort to extend the gas mileage formula

$$D = RT, \qquad \textbf{Distance = Rate} \times \textbf{Time},$$

to instantaneous rates.

| Definite Integrals | Indefinite Integrals |
|---|---|
| $(a)\quad \int_a^b f'(x)dx = f(b) - f(a)$ | $\int y'(x)dx = y(x) + C$ |
| $(b)\quad \left(\int_a^x g(t)dt\right)' = g(x)$ | $\left(\int g(x)dx\right)' = g(x)$ |

Part (a) is used in differential equations $y' = f(x, y)$ to discover the solution $y(x)$. Part (b) is used in checking the answer.

## Method of Quadrature

Also called the *integration method*, the idea is to multiply the differential equation by $dx$, then write an integral sign on each side. The logic is that equal integrands imply equal integrals.

- The method applies only to quadrature equations $y' = F(x)$.

  **Quadrature Classification Test**
  The equation $y' = f(x,y)$ is a quadrature equation if and only if function $f(x,y)$ is independent of $y$, or equivalently, $\frac{\partial f}{\partial y} = 0$.

- The Fundamental Theorem of Calculus is applied on the left side to evaluate $\int y'(x)dx = y(x) + C$, where $C$ is a constant.
- The method finds a candidate solution $y(x)$. It does not verify that the expression works.

## Example: Method of Quadrature

**1 Example (Quadrature)** Solve $y' = 2x$ by the method of quadrature.

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- Multiply $y' = 2x$ by $dx$, then write an integral sign on each side.

$$\int y'(x)dx = \int 2x dx$$

- Apply the FTC $\int y'(x)dx = y(x) + C$ on the left:

$$y(x) + c_1 = \int 2x dx$$

- Evaluate the integral on the right by tables. Then

$$y(x) + c_1 = x^2 + c_2, \quad \text{or} \quad y(x) = x^2 + C$$

**2 Example (Quadrature)** Solve $y' = 3e^x$, $y(0) = 2$.

**Candidate solution**. The *method of quadrature* is applied.

| | |
|---|---|
| $y'(x) = 3e^x$ | Copy the equation. |
| $y'(x)dx = 3e^x dx$ | Multiply both sides by $dx$. |
| $\int y'(x)dx = \int 3e^x dx$ | Add an integral on each side. |
| $y(x) + c_1 = 3e^x + c_2$ | Fundamental theorem of calculus (FTC) used left. Integral table used right. |
| $y(x) = 3e^x + C$ | Quadrature complete. Next, find $C$. |
| $2 = y(0) = 3e^0 + C$ | Substitute $x = 0$. Use $y(0) = 2$. |
| $y(x) = 3e^x - 1$ | Substitute $C = -1$. Solution candidate found. |

**Candidate Solution:** $\boxed{y(x) = 3e^x - 1}$

## Two-Panel Answer Check

A typical answer check involves two panels, because two equations must be tested: (1) The differential equation, and (2) The initial condition. Abbreviations LHS=*Left-Hand-side* and RHS=*Right-Hand-Side* are used in the displays.

**Verify DE**. Panel 1 of the answer check tests the solution $y = 3e^x - 1$ of the differential equation (DE) $y' = 3e^x$:

| | | |
|---|---|---|
| LHS | $= y'$ | Left side of the differential equation. |
| | $= (3e^x - 1)'$ | Substitute $y = 3e^x - 1$. |
| | $= 3e^x - 0$ | Sum rule, constant rule and $(e^u)' = u'e^u$. |
| | $=$ RHS | DE verified. |

**Verify IC**. Panel 2 of the answer check tests the initial condition (IC) $y(0) = 2$:

| | | |
|---|---|---|
| LHS | $= y(0)$ | Left side of the initial condition $y(0) = 2$. |
| | $= (3e^x - 1)\|_{x=0}$ | Substitute $y = 3e^x - 1$. |
| | $= 3e^0 - 1$ | |
| | $= 2$ | Simplify using $e^0 = 1$. |
| | $=$ RHS | IC verified. |