2.2 Separable Equations

An equation $y' = f(x, y)$ is called **separable** provided algebraic operations, usually multiplication, division and factorization, allow it to be written in a **separable form** $y' = F(x)G(y)$ for some functions F and G. This class includes the *quadrature equations* $y' = F(x)$. Separable equations and associated solution methods were discovered by G. Leibniz in 1691 and formalized by J. Bernoulli in 1694.

Finding a Separable Form

Given differential equation $y' = f(x, y)$, invent values x_0, y_0 such that $f(x_0, y_0) \neq 0$. Define F, G by the formulas

$$F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \quad G(y) = f(x_0, y). \tag{1}$$

Because $f(x_0, y_0) \neq 0$, then (1) makes sense. Test I below implies the following test.

Theorem 2 (Separability Test)
Let F and G be defined by equation (1). Compute $F(x)G(y)$. Then

(a) $F(x)G(y) = f(x, y)$ implies $y' = f(x, y)$ is **separable**.

(b) $F(x)G(y) \neq f(x, y)$ implies $y' = f(x, y)$ is **not separable**.

Invention and Application. Initially, let (x_0, y_0) be $(0, 0)$ or $(1, 1)$ or some suitable pair, for which $f(x_0, y_0) \neq 0$; then define F and G by (1). Multiply to test the equation $FG = f$. The algebra will discover a factorization $f = F(x)G(y)$ without having to know algebraic tricks like factorizing multi-variable equations. But if $FG \neq f$, then the algebra *proves* the equation is not separable.

Non-Separability Tests

Test I — Equation $y' = f(x, y)$ is not separable if for some pair of points (x_0, y_0), (x, y) in the domain of f

$$f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0. \tag{2}$$

Test II — The equation $y' = f(x, y)$ is not separable if either $f_x(x, y)/f(x, y)$ is non-constant in y or $f_y(x, y)/f(x, y)$ is non-constant in x.

Illustration. Consider $y' = xy + y^2$. *Test I* implies it is not separable, because $f(x, 1)f(0, y) - f(0, 1)f(x, y) = (x+1)y^2 - (xy+y^2) = x(y^2 - y) \neq 0$. *Test II* implies it is not separable, because $f_x/f = 1/(x + y)$ is not constant as a function of y.

Test I details. Assume $f(x,y) = F(x)G(y)$, then equation (2) fails because each term on the left side of (2) evaluates to $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y) (hence contradiction $0 \neq 0$).

Test II details. Assume $f(x,y) = F(x)G(y)$ and F, G are sufficiently differentiable. Then $f_x(x,y)/f(x,y) = F'(x)/F(x)$ is independent of y and $f_y(x,y)/f(x,y) = G'(y)/G(y)$ is independent of x.

Separated Form Test

A **separated equation** $y'/G(y) = F(x)$ is recognized by this test:

> **Left Side Test.** The left side of the equation has factor y' and it is independent of symbol x.
>
> **Right Side Test.** The right side of the equation is independent of symbols y and y'.

Variables-Separable Method

Determined by the method are the following kinds of solution formulas.

Equilibrium Solutions. They are the constant solutions $y = c$ of $y' = f(x,y)$. Find them by substituting $y = c$ in $y' = f(x,y)$, followed by solving for c, then report the list of answers $y = c$ for all values of c.

Non-Equilibrium Solutions. For separable equation $y' = F(x)G(y)$, it is a solution y with $G(y) \neq 0$. It is found by dividing by $G(y)$ and applying the method of quadrature.

The term *equilibrium* is borrowed from kinematics. Alternative terms are **rest solution** and **stationary solution**; all mean $y' = 0$ in calculus terms.

Spurious Solutions. If $F(x)G(y) = 0$ is solved instead of $G(y) = 0$, then both x and y solutions might be found. The x-solutions are ignored: they are not equilibrium solutions. Only solutions of the form $y =$ constant are called equilibrium solutions.

It is important to *check the solution* to a separable equation, because certain steps used to arrive at the solution may not be reversible.

For most applications, the two kinds of solutions suffice to determine all possible solutions. In general, a separable equation may have non-unique solutions to some initial value problem. To prevent this from happening, it can be assumed that F, G and G' are continuous; see the Picard-Lindelöf theorem, page 62. If non-uniqueness does occur, then often the equilibrium and non-equilibrium solutions can be pieced together to represent all solutions.

Finding Equilibria

The search for equilibria can be done without finding the separable form of $y' = f(x, y)$. It is enough to solve for y in the equation $f(x, y) = 0$, *subject to the condition that x is arbitrary.* An equilibrium solution y cannot depend upon x, because it is *constant.* If y turns out to depend on x, after solving $f(x, y) = 0$ for y, then this is sufficient evidence that $y' = f(x, y)$ is **not separable**. Some examples:

$y' = y\sin(x - y)$	It is *not separable*. The solutions of $y\sin(x-y) = 0$ are $y = 0$ and $x-y = n\pi$ for any integer n. The solution $y = x-n\pi$ is non-constant, therefore the equation cannot be separable.
$y' = xy(1 - y^2)$	It is *separable*. The equation $xy(1 - y^2) = 0$ has three constant solutions $y = 0$, $y = 1$, $y = -1$.

The problem of finding equilibria is known to be technically unsolvable, that is, there is no proven algorithm for finding all the solutions of $G(y) = 0$. However, there are some very good numerical methods that apply, including **Newton's method** and the **bisection method**. Modern computer algebra systems make it practical to find both algebraic and numerical equilibrium solutions, in a single effort.

Finding Non-Equilibrium Solutions

A given solution $y(x)$ satisfying $G(y(x)) \neq 0$ throughout its domain of definition is called a non-equilibrium solution. Then division by $G(y(x))$ is allowed. The *method of quadrature* applies to the separated equation $y'/G(y(x)) = F(x)$. Some details:

$\int_{x_0}^{x} \dfrac{y'(t)dt}{G(y(t))} = \int_{x_0}^{x} F(t)dt$	Integrate both sides of the separated equation over $x_0 \le t \le x$.
$\int_{y_0}^{y(x)} \dfrac{du}{G(u)} = \int_{x_0}^{x} F(t)dt$	Apply on the left the change of variables $u = y(t)$. Define $y_0 = y(x_0)$.
$y(x) = W^{-1}\left(\int_{x_0}^{x} F(t)dt\right)$	Define $W(y) = \int_{y_0}^{y} du/G(u)$. Take inverses to isolate $y(x)$.

The calculation produces a formula which is strictly speaking a *candidate solution y*. It does not prove that the formula works in the equation: *checking the solution* is required.

Theoretical Inversion

The function W^{-1} appearing in the last step above is generally not given by a formula. Therefore, W^{-1} rarely appears explicitly in applications or examples. It is the *method* that is memorized:

> Prepare a separable differential equation by transforming it to separated form. Then apply the method of quadrature.

The theoretical basis for using W^{-1} is a calculus theorem which says that *a strictly monotone continuous function has a continuous inverse*. The fundamental theorem of calculus implies that $W(y)$ is continuous with nonzero derivative $W'(y) = 1/G(y)$. Therefore, $W(y)$ is strictly monotone. The cited calculus theorem implies that $W(y)$ has a continuously differentiable inverse W^{-1}.

Explicit and Implicit Solutions

The variables-separable method gives equilibrium solutions which are already *explicit*, that is:

Definition 1 (Explicit Solution)
A solution of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y.

Definition 2 (Implicit Solution)
A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

Equations like $2y = x$ are not explicit (they are called *implicit*) because the coefficient of y on the left is not 1. Similarly, $y = x + y^2$ is not explicit because the right side contains symbol y. Equation $y = e^\pi$ is explicit because the right side fails to contain symbol y (symbol x may be absent). Applications can leave the non-equilibrium solutions in *implicit* form $\int_{y_0}^{y(x)} du/G(u) = \int_{x_0}^{x} F(t)dt$, with serious effort being expended to do the indicated integrations.

In special cases, it is possible to find an explicit solution from the implicit one by algebraic methods. Students find the algebraic methods to be unmotivated tricks. Computer algebra systems can make this step look like science instead of art.

Examples

3 **Example (Non-separable Equation)** Explain why $yy' = x - y^2$ is not separable.

Solution: It is tempting to try manipulations like adding y^2 to both sides of the equation, in an attempt to obtain a separable form, but every such trick fails. The failure of such attempts is evidence that the equation is perhaps not separable. Failure of attempts does not *prove* non-separability.

Test I applies to verify that the equation is not separable. Let $f(x,y) = x/y - y$ and choose $x_0 = 0$, $y_0 = 1$. Then $f(x_0, y_0) \neq 0$. Compute as follows:

$\text{LHS} = f(x,y_0)f(x_0,y) - f(x_0,y_0)f(x,y)$	Identity (2) left side.
$= f(x,1)f(0,y) - f(0,1)f(x,y)$	Use $x_0 = 0$, $y_0 = 1$.
$= (x-1)(-y) - (-1)(x/y - y)$	Substitute $f(x,y) = x/y - y$.
$= -xy + x/y$	Simplify.

This expression fails to be zero for all (x, y) (e.g., $x = 1$, $y = 2$), therefore the equation is not separable, by *Test I*.

Test II also applies to verify the equation is not separable: $f_x/f = 1/yf = x - y^2$ is non-constant in x.

4 Example (Separated Form Test Failure) Given $yy' = 1 - y^2$, explain why the equivalent equation $yy' + y^2 = 1$, obtained by adding y^2 across the equation, fails the separated form test, page 75.

Solution: The *test* requires the left side of $yy' + y^2 = 1$ to contain the factor y'. It doesn't, so it fails the test. Yes, $yy' + y^2 = 1$ does pass the other checkpoints of the *test*: the left side is independent of x and the right side is independent of y and y'.

5 Example (Separated Equation) Find for $(x+1)yy' = x - xy^2$ a separated equation using the *test*, page 75.

Solution: The equation usually reported is $\dfrac{yy'}{(1-y)(1+y)} = \dfrac{x}{x+1}$. It is found by factoring and division.

The given equation is factored into $(1 + x)yy' = x(1 - y)(1 + y)$. To pass the *test*, the objective is to move all factors containing only y to the left and all factors containing only x to the right. This is technically accomplished using division by $(x + 1)(1 - y)(1 + y)$.

To the result of the division is applied the *test* on page 75: the *left side* contains factor y' and otherwise involves the factor $y/(1 - y^2)$, which depends only on y; the *right side* is $x/(x + 1)$, which depends only on x. In short, the candidate separated equation passes the test.

There is another way to approach the problem, by writing the differential equation in standard form $y' = f(x,y)$ where $f(x,y) = x(1 - y^2)/(1 + x)$. Then $f(1, 0) = 1/2 \neq 0$. Define $F(x) = f(x, 0)/f(1, 0)$, $G(y) = f(1, y)$. We verify $F(x)G(y) = f(x, y)$. A separated form is then $y'/G(y) = F(x)$ or $2y'/(1 - y^2) = 2x/(1 + x)$.

6 Example (Equilibria) Given $y' = x(1 - y)(1 + y)$, find all equilibria.

Solution: The constant solutions $y = -1$ and $y = 1$ are the equilibria, as will be justified.

Equilibria are found by substituting $y = c$ into the differential equation $y' = x(1-y)(1+y)$, which gives the equation

$$x(1-c)(1+c) = 0.$$

The formal college algebra solutions are $x = 0$, $c = -1$ and $c = 1$. However, we do not seek these college algebra answers! Equilibria are the solutions $y = c$ such that $x(1-c)(1+c) = 0$ *for all* x. The conditional *for all* x causes the algebra problem to reduce to just two equations: $0 = 0$ (from $x = 0$) and $(1-c)(1+c) = 0$ (from $x \neq 0$). We solve for $c = 1$ and $c = -1$, then report the two equilibrium solutions $y = 1$ and $y = -1$.

7 **Example (Non-Equilibria)** Given $y' = x^2(1+y)$, $y(0) = y_0$, find all non-equilibrium solutions.

Solution: The unique solution is $y = (1+y_0)e^{x^3/3} - 1$. Details follow.

The separable form $y' = F(x)G(y)$ is realized for $F(x) = x^2$ and $G(y) = 1 + y$. Sought are solutions with $G(y) \neq 0$, or simply $1 + y \neq 0$.

$y' = x^2(1+y)$	Original equation.
$\dfrac{y'}{1+y} = x^2$	Divide by $1 + y$. Separated form found.
$\int \dfrac{y'}{1+y} dx = \int x^2 dx$	Method of quadrature.
$\int \dfrac{du}{1+u} = \int x^2 dx$	Change variables $u = y(x)$ on the left.
$\ln\lvert 1+y(x)\rvert = x^3/3 + c$	Evaluate integrals. Implicit solution found.

Applications might stop at this point and report the *implicit solution*. This illustration continues, to find the *explicit solution* $y = (1+y_0)e^{x^3/3} - 1$.

$\lvert 1+y(x)\rvert = e^{x^3/3+c}$	By definition, $\ln u = w$ means $u = e^w$.
$1 + y(x) = ke^{x^3/3+c}$	Drop absolute value, $k = \pm 1$.
$y(x) = ke^{x^3/3+c} - 1$	Candidate solution. Constants unresolved.

The initial condition $y(0) = y_0$ is used to resolve the constants c and k. First, $|1+y_0| = e^c$ from the first equation. Second, $1+y_0$ and $1+y(x)$ must have the same sign (they are never zero), so $k(1+y_0) > 0$. Hence, $1 + y_0 = ke^c$, which implies the solution is $y = ke^c e^{x^3/3} - 1$ or $y = (1+y_0)e^{x^3/3} - 1$.

8 **Example (Equilibria)** Given $y' = x\sin(1-y)\cos(1+y)$, find all equilibrium solutions.

Solution: The infinite set of equilibria are justified below to be

$$y = 1 + n\pi, \quad y = -1 + (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

A separable form $y' = F(x)G(y)$ is verified by defining $F(x) = x$ and $G(y) = \sin(1-y)\cos(1+y)$. Equilibria $y = c$ are found by solving for c in the equation $G(c) = 0$, which is

$$\sin(1-c)\cos(1+c) = 0.$$

This equation is satisfied when the argument of the sine is an integer multiple of π or when the argument of the cosine is an odd integer multiple of $\pi/2$. The solutions are $c - 1 = 0, \pm\pi, \pm 2\pi, \ldots$ and $1 + c = \pm\pi/2, \pm 3\pi/2, \ldots$.

Multiple solutions and `maple`. Equations having multiple solutions may require **CAS** setup. Below, the first code fragment returns two solutions, $y = 1$ and $y = -1 + \pi/2$. The second returns all solutions.

```
# The default returns two solutions
G:=y->sin(1-y)*cos(1+y):
solve(G(y)=0,y);
# Special setup returns all solutions
_EnvAllSolutions := true:
G:=y->sin(1-y)*cos(1+y):
solve(G(y)=0,y);
```

9 Example (Non-Equilibria) Given $y' = x^2\sin(y)$, $y(0) = y_0$, justify that all non-equilibrium solutions are given by[1]

$$y = 2\text{Arctan}\left(\tan(y_0/2)e^{x^3/3}\right) + 2n\pi.$$

Solution: A separable form $y' = F(x)G(y)$ is defined by $F(x) = x^2$ and $G(y) = \sin(y)$. A non-equilibrium solution will satisfy $G(y) \neq 0$, or simply $\sin(y) \neq 0$. Define n by $y_0/2 = \text{Arctan}(\tan(y_0/2)) + n\pi$, where $|\text{Arctan}(u)| < \pi/2$. Then

$y' = x^2\sin(y)$	The original equation.
$\csc(y)y' = x^2$	Separated form. Divided by $\sin(y) \neq 0$.
$\int \csc(y)y'dx = \int x^2dx$	Quadrature using indefinite integrals.
$\int \csc(u)du = \int x^2dx$	Change variables $u = y(x)$ on the left.
$\ln\lvert\csc y(x) - \cot y(x)\rvert = \frac{1}{3}x^3 + c$	Integral tables. Implicit solution found.

Trigonometric Identity. Integral tables make use of the identity $\tan(y/2) = \csc y - \cot y$, which is derived from the relations $2\theta = y$, $1 - \cos 2\theta = 2\sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$. Tables offer an alternate answer for the last integral above, $\ln|\tan(y/2)|$.

The solution obtained at this stage is called an *implicit solution*, because y has not been isolated. It is possible to solve for y in terms of x, an *explicit solution*. The details:

$\lvert\csc y - \cot y\rvert = e^{x^3/3+c}$	By definition, $\ln u = w$ means $u = e^w$.
$\csc y - \cot y = ke^{x^3/3+c}$	Assign $k = \pm 1$ to drop absolute values.

[1]While $\theta = \arctan u$ gives any angle, $\theta = \text{Arctan}(u)$ gives $|\theta| < \pi/2$.

$\dfrac{1-\cos y}{\sin y} = ke^{x^3/3+c}$	Then k has the same sign as $\sin(y)$, because $1-\cos y \geq 0$.
$\tan(y/2) = ke^{x^3/3+c}$	Use $\tan(y/2) = \csc y - \cot y$.
$y = 2\text{Arctan}\left(ke^{x^3/3+c}\right) + 2n\pi$	Candidate solution, $n = 0, \pm1, \pm2, \ldots$

Resolving the Constants. Constants c and k are uniquely resolved for a given initial condition $y(0) = y_0$. Values $x = 0$ and $y = y_0$ determine constant c by the equation $\tan(y_0/2) = ke^c$ (two equations back). The condition $k\sin(y_0) > 0$ determines k, because $\sin y_0$ and $\sin y$ have identical signs. If n is defined by $y_0/2 = \text{Arctan}(\tan(y_0/2)) + n\pi$ and $K = ke^c = \tan(y_0/2)$, then the *explicit solution* is

$$y = 2\text{Arctan}\left(Ke^{x^3/3}\right) + 2n\pi, \quad K = \tan(y_0/2).$$

Trigonometric identities and `maple`. Using the identity $\csc y - \cot y = \tan(y/2)$, `maple` finds the same relation. Complications occur without it.

```
_EnvAllSolutions := true:
solve(csc(y)-cot(y)=k*exp(x^3/3+c),y);
solve(tan(y/2)=k*exp(x^3/3+c),y);
```

10 Example (Independent of x) Solve $y' = y(1 - \ln y)$, $y(0) = y_0$.

Solution: There is just one equilibrium solution $y = e \approx 2.718$. Not included is $y = 0$, because $y(1 - \ln y)$ is undefined for $y \leq 0$. Details appear below for the explicit solution (which includes $y = e$)

$$y = e^{1 - (1 - \ln y_0)e^{-x}}.$$

An equation $y' = f(x, y)$ independent of x has the form $y' = F(x)G(y)$ where $F(x) = 1$. Divide by $G(y)$ to obtain a separated form $y'/G(y) = 1$. In the present case, $G(y) = y(1 - \ln y)$ is defined for $y > 0$. To require $G(y) \neq 0$ means $y > 0$, $y \neq e$. Non-equilibrium solutions will satisfy $y > 0$ and $y \neq e$.

$\dfrac{y'}{y(1-\ln y)} = 1$	Separated form. Assume $y > 0$ and $y \neq e$.
$\int \dfrac{y'}{y(1-\ln y)} dx = \int dx$	Method of quadrature.
$\int \dfrac{-du}{u} = \int dx$	Substitute $u = 1 - \ln y$ on the left. Chain rule $(\ln y)' = y'/y$ applied; $du = -y'dx/y$.
$-\ln\lvert 1 - \ln y(x)\rvert = x + c$	Evaluate the integral using $u = 1 - \ln y$. Implicit solution found.

The remainder of the solution contains college algebra details, to find from the *implicit solution* all *explicit solutions* y.

$\lvert 1 - \ln y(x)\rvert = e^{-x-c}$	Use $\ln u = w$ equivalent to $u = e^w$.
$1 - \ln y(x) = ke^{-x-c}$	Drop absolute value, $k = \pm1$.
$\ln y(x) = 1 - ke^{-x-c}$	Solved for $\ln y$.

$y(x) = e^{1 - ke^{-x-c}}$ Candidate solution; c and k unresolved.

To resolve the constants, start with $y_0 > 0$ and $y_0 \neq e$. To determine k, use the requirement $G(y) \neq 0$ to deduce that $k(1 - \ln y(x)) > 0$. At $x = 0$, it means $k|1 - \ln y_0| = 1 - \ln y_0$. Then $k = 1$ for $0 < y_0 < e$ and $k = -1$ otherwise.

Let $y = y_0$, $x = 0$ to determine c through the equation $|1 - \ln y_0| = e^{-c}$. Combining with the value of k gives $1 - \ln y_0 = ke^{-c}$.

Assembling the answers for k and c produces the relations

$y = e^{1 - ke^{-x-c}}$ Candidate solution.

$= e^{1 - ke^{-c}e^{-x}}$ Exponential rule $e^{a+b} = e^a e^b$.

$= e^{1 - (1 - \ln y_0)e^{-x}}$ Explicit solution. Used $ke^{-c} = 1 - \ln y_0$.

Even though the solution has been found through legal methods, it remains to *verify the solution*. See the exercises.

Exercises 2.2

Separated Form Test. Test the given equation by the separated form test on page 75.

Report whether or not the equation *passes* or *fails*.

In this test, it is not allowed to do algebraic operations on the equation – test the equation as written. See also Examples 3 and 4, page 77.

1. $y' = 2$

2. $y' = x$

3. $y' + y = 2$

4. $y' + 2y = x$

5. $yy' = 2 - x$

6. $2yy' = x + x^2$

7. $yy' + \sin(y') = 2 - x$

8. $2yy' + \cos(y) = x$

9. $2yy' = y'\cos(y) + x$

10. $(2y + \tan(y))y' = x$

Separated Equation. Determine the separated form $y'/G(y) = F(x)$ for the given separable equation. See Example 5, page 78.

11. $(1 + x)y' = 2 + y$

12. $(1 + y)y' = xy$

13. $y' = \dfrac{x + xy}{(x + 1)^2 - 1}$

14. $y' = \sin(x)\dfrac{1 + y}{(x + 2)^2 - 4}$

15. $xy' = y\sin(y)\cos(x)$

16. $x^2y' = y\cos(y)\tan(x)$

17. $y^2(x - y)y' = \dfrac{x^2 - y^2}{x + y}$

18. $xy^2(x + y)y' = \dfrac{y^2 - x^2}{x - y}$

19. $xy^2y' = \dfrac{y - x}{x - y}$

20. $xy^2y' = \dfrac{x^2 - xy}{x - y}$

Equilibrium solutions. Determine the equilibria for the given equation. See Examples 6 and 8.

21. $y' = xy(1 + y)$

22. $xy' = y(1-y)$

23. $y' = \dfrac{1+y}{1-y}$

24. $xy' = \dfrac{y(1-y)}{1+y}$

25. $y' = (1+x)\tan(y)$

26. $y' = y(1+\ln y)$

27. $y' = xe^y(1+y)$

28. $xy' = e^y(1-y)$

29. $xy' = e^y(1-y^2)(1+y)^3$

30. $xy' = e^y(1-y^3)(1+y^3)$

Non-Equilibrium Solutions. Find the non-equilibrium solutions for the given separable equation. See Examples 7 and 9 for details.

31. $y' = (xy)^{1/3}$, $y(0) = y_0$.

32. $y' = (xy)^{1/5}$, $y(0) = y_0$.

33. $y' = 1 + x - y - xy$, $y(0) = y_0$.

34. $y' = 1 + x + 2y + 2xy$, $y(0) = y_0$.

35. $y' = \dfrac{(x+1)y^3}{x^2(y^3-y)}$, $y(0) = y_0$.

36. $y' = \dfrac{(x-1)y^2}{x^3(y^3+y)}$, $y(0) = y_0$.

37. $2yy' = x(1-y^2)$

38. $2yy' = x(1+y^2)$

39. $(1+x)y' = 1-y$, $y(0) = y_0$.

40. $(1-x)y' = 1+y$, $y(0) = y_0$.

41. $\tan(x)y' = y$, $y(\pi/2) = y_0$.

42. $\tan(x)y' = 1+y$, $y(\pi/2) = y_0$.

43. $\sqrt{x}y' = \cos^2(y)$, $y(1) = y_0$.

44. $\sqrt{1-x}y' = \sin^2(y)$, $y(0) = y_0$.

45. $\sqrt{x^2-16}yy' = x$, $y(5) = y_0$.

46. $\sqrt{x^2-1}yy' = x$, $y(2) = y_0$.

47. $y' = x^2(1+y^2)$, $y(0) = 1$.

48. $(1-x)y' = x(1+y^2)$, $y(0) = 1$.

Independent of x. Solve the given equation, finding all solutions. See Example 10.

49. $y' = \sin y$, $y(0) = y_0$.

50. $y' = \cos y$, $y(0) = y_0$.

51. $y' = y(1+\ln y)$, $y(0) = y_0$.

52. $y' = y(2+\ln y)$, $y(0) = y_0$.

53. $y' = y(y-1)(y-2)$, $y(0) = y_0$.

54. $y' = y(y-1)(y+1)$, $y(0) = y_0$.

55. $y' = y^2 + 2y + 5$, $y(0) = y_0$.

56. $y' = y^2 + 2y + 7$, $y(0) = y_0$.

Details in the Examples. Collected here are verifications for details in the examples. Brief solutions for odd exercises appear in the *Answers*.

57. **(Example 6)** The equation $x(1-y)(1+y) = 0$ was solved in the example, but $x = 0$ was ignored, and only $y = -1$ and $y = 1$ were reported. Why?

58. **(Example 7)** An absolute value equation $|u| = w$ was replaced by $u = kw$ where $k = \pm 1$. Justify the replacement using the *definition* $|u| = u$ for $u \ge 0$, $|u| = -u$ for $u < 0$.

59. **(Example 7)** Verify directly that $y = (1+y_0)e^{x^3/3} - 1$ solves the initial value problem $y' = x^2(1+y)$, $y(0) = y_0$.

60. **(Example 8)** The relation $y = 1 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ describes the list $\ldots, 1-\pi, 1, 1+\pi, \ldots$. Write the list for the relation $y = -1 + (2n+1)\frac{\pi}{2}$.

61. **(Example 8)** Solve $\sin(u) = 0$ and $\cos(v) = 0$ for u and v. Supply graphs which show why there are infinity many solutions.

62. **(Example 9)** Explain why $y_0/2$ does not equal $\text{Arctan}(\tan(y_0/2))$. Give a calculator example.

63. **(Example 9)** Establish the identity $\tan(y/2) = \csc y - \cot y$.

64. **(Example 10)** Let $y_0 > 0$. Verify that $y = e^{1 - (1 - \ln y_0)e^{-x}}$ solves

$$y' = y(1 - \ln y), \quad y(0) = y_0.$$