Basic Laplace Theory

- Laplace Integral
- Direct Laplace Transform
- A First LaPlace Table
- A Minimal LaPlace Table
- Forward LaPlace Table
- Backward LaPlace Table
- Some Transform Rules
- Lerch's Cancelation Law and the Fundamental Theorem of Calculus
- Illustration in Calculus Notation
- Illustration Translated to Laplace $\boldsymbol{L}$-notation

Laplace Integral

The integral

$$\int_0^\infty g(t)e^{-st}dt$$

is called the **Laplace integral** of the function $g(t)$. It is defined by

$$\int_0^\infty g(t)e^{-st}dt \equiv \lim_{N\to\infty}\int_0^N g(t)e^{-st}dt$$

and it depends on variable s. The ideas will be illustrated for $g(t) = 1$, $g(t) = t$ and $g(t) = t^2$. Results appear in Table 1 *infra*.

Laplace Integral or Direct Laplace Transform

The **Laplace integral** or the **direct Laplace transform** of a function $f(t)$ defined for $0 \le t < \infty$ is the ordinary calculus integration problem

$$\int_0^\infty f(t)e^{-st}dt.$$

The *Laplace integrator* is $dx = e^{-st}dt$ instead of the usual dt.

A Laplace integral is succinctly denoted in science and engineering literature by the symbol

$$L(f(t)),$$

which abbreviates

$$\int_E (f(t))dx,$$

with set $E = [0, \infty)$ and Laplace integrator $dx = e^{-st}dt$.

A First LaPlace Table

$\int_0^\infty (1)e^{-st}dt = -(1/s)e^{-st}\Big\vert_{t=0}^{t=\infty}$	Laplace integral of $g(t) = 1$.
$= 1/s$	Assumed $s > 0$.
$\int_0^\infty (t)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(e^{-st})dt$	Laplace integral of $g(t) = t$.
$= -\frac{d}{ds}\int_0^\infty (1)e^{-st}dt$	Use $\int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt$.
$= -\frac{d}{ds}(1/s)$	Use $L(1) = 1/s$.
$= 1/s^2$	Differentiate.
$\int_0^\infty (t^2)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(te^{-st})dt$	Laplace integral of $g(t) = t^2$.
$= -\frac{d}{ds}\int_0^\infty (t)e^{-st}dt$	
$= -\frac{d}{ds}(1/s^2)$	Use $L(t) = 1/s^2$.
$= 2/s^3$	

Summary

Table 1. Laplace integral $\int_0^\infty g(t)e^{-st}dt$ for $g(t) = 1, t$ and t^2.

$$\int_0^\infty (1)e^{-st}\,dt = \frac{1}{s}, \qquad \int_0^\infty (t)e^{-st}\,dt = \frac{1}{s^2}, \qquad \int_0^\infty (t^2)e^{-st}\,dt = \frac{2}{s^3}.$$

In summary, $\quad L(t^n) = \dfrac{n!}{s^{1+n}}$

A Minimal Laplace Table

Solving differential equations by Laplace methods requires keeping a smallest table of Laplace integrals available, usually memorized. The last three entries will be verified later.

Table 2. A minimal Laplace integral table with L-notation

$\int_0^\infty (t^n)e^{-st}\,dt = \dfrac{n!}{s^{1+n}}$	$L(t^n) = \dfrac{n!}{s^{1+n}}$
$\int_0^\infty (e^{at})e^{-st}\,dt = \dfrac{1}{s-a}$	$L(e^{at}) = \dfrac{1}{s-a}$
$\int_0^\infty (\cos bt)e^{-st}\,dt = \dfrac{s}{s^2+b^2}$	$L(\cos bt) = \dfrac{s}{s^2+b^2}$
$\int_0^\infty (\sin bt)e^{-st}\,dt = \dfrac{b}{s^2+b^2}$	$L(\sin bt) = \dfrac{b}{s^2+b^2}$

Forward Laplace Table

The forward table finds the Laplace integral $L(f(t))$ when $f(t)$ is a linear combinations of atoms. The Laplace calculus rules apply to find the Laplace integral of $f(t)$ when it is not in this short table.

Table 3. Forward Laplace integral table

Function $f(t)$	Laplace Integral $L(f(t))$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{1+n}}$
e^{at}	$\frac{1}{s-a}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$

Backward Laplace Table

The backward table finds $f(t)$ from a Laplace integral $L(f(t))$ expression. Always, $f(t)$ is a linear combinations of atoms. The Laplace calculus rules apply to find $f(t)$ when it is does not appear in this short table.

Table 4. Backward Laplace integral table

Laplace Integral $L(f(t))$	$f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^{1+n}}$	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+b^2}$	$\cos bt$
$\frac{1}{s^2+b^2}$	$\frac{\sin bt}{b}$

Some Transform Rules

$L(f(t) + g(t)) = L(f(t)) + L(g(t))$	The integral of a sum is the sum of the integrals.
$L(cf(t)) = cL(f(t))$	Constants c pass through the integral sign.
$L(y'(t)) = sL(y(t)) - y(0)$	The t-derivative rule, or integration by parts.

Lerch's Cancelation Law and the Fundamental Theorem of Calculus

$$L(y(t)) = L(f(t)) \text{ implies } y(t) = f(t) \quad \text{Lerch's cancelation law.}$$

Lerch's cancelation law in integral form is

$$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \quad \text{implies} \quad y(t) = f(t). \tag{1}$$

Quadrature Methods

Lerch's Theorem is used *last* in Laplace's quadrature method. In Newton calculus, the quadrature method uses the Fundamental Theorem of Calculus *first*. The two theorems have a similar use, to *isolate* the solution y of the differential equation.

An illustration

Laplace's method will be applied to solve the initial value problem

$$\boldsymbol{y' = -1, \quad y(0) = 0.}$$

Illustration Details

Table 5. Laplace method details for $y' = -1, y(0) = 0$.

$y'(t)e^{-st}dt = -e^{-st}dt$	Multiply $y' = -1$ by $e^{-st}dt$.
$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$	Integrate $t = 0$ to $t = \infty$.
$\int_0^\infty y'(t)e^{-st}dt = -1/s$	Use Table 1.
$s\int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$	Integrate by parts on the left.
$\int_0^\infty y(t)e^{-st}dt = -1/s^2$	Use $y(0) = 0$ and divide.
$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$	Use Table 1.
$y(t) = -t$	Apply Lerch's cancelation law.

Translation to L-notation

Table 6. Laplace method L-notation details for $y' = -1$, $y(0) = 0$ translated from Table 5.

$L(y'(t)) = L(-1)$	Apply L across $y' = -1$, or multiply $y' = -1$ by $e^{-st}dt$, integrate $t = 0$ to $t = \infty$.
$L(y'(t)) = -1/s$	Use Table 1 forwards.
$sL(y(t)) - y(0) = -1/s$	Integrate by parts on the left.
$L(y(t)) = -1/s^2$	Use $y(0) = 0$ and divide.
$L(y(t)) = L(-t)$	Apply Table 1 backwards.
$y(t) = -t$	Invoke Lerch's cancelation law.

1 Example (Laplace method) Solve by Laplace's method the initial value problem $y' = 5 - 2t$, $y(0) = 1$ to obtain $y(t) = 1 + 5t - t^2$.

Solution: Laplace's method is outlined in Tables 5 and 6. The L-notation of Table 6 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$L(y'(t)) = L(5 - 2t)$	Apply L across $y' = 5 - 2t$.
$\quad = 5L(1) - 2L(t)$	Linearity of the transform.
$\quad = \dfrac{5}{s} - \dfrac{2}{s^2}$	Use Table 1 forwards.
$sL(y(t)) - y(0) = \dfrac{5}{s} - \dfrac{2}{s^2}$	Apply the t-derivative rule.
$L(y(t)) = \dfrac{1}{s} + \dfrac{5}{s^2} - \dfrac{2}{s^3}$	Use $y(0) = 1$ and divide.
$L(y(t)) = L(1) + 5L(t) - L(t^2)$	Use Table 1 backwards.
$\quad = L(1 + 5t - t^2)$	Linearity of the transform.
$y(t) = 1 + 5t - t^2$	Invoke Lerch's cancelation law.

2 Example (Laplace method) Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$ to obtain $y(t) = 5t^2$.

Solution: The L-notation of Table 6 will be used to find the solution $y(t) = 5t^2$.

$L(y''(t)) = L(10)$	Apply L across $y'' = 10$.
$sL(y'(t)) - y'(0) = L(10)$	Apply the t-derivative rule to y'.
$s[sL(y(t)) - y(0)] - y'(0) = L(10)$	Repeat the t-derivative rule, on y.
$s^2L(y(t)) = 10L(1)$	Use $y(0) = y'(0) = 0$.
$L(y(t)) = \dfrac{10}{s^3}$	Use Table 1 forwards. Then divide.
$L(y(t)) = L(5t^2)$	Use Table 1 backwards.
$y(t) = 5t^2$	Invoke Lerch's cancelation law.