Systems of Differential Equations and Laplace's Method

- Solving $\boldsymbol{x}' = C\boldsymbol{x}$
- The Resolvent
- An Illustration for $\mathbf{x}' = C\mathbf{x}$

Solving $x' = Cx$

Apply L to each side to obtain $L(\mathrm{x}') = CL(\mathrm{x})$. Use the parts rule

$$L(\mathrm{x}') = sL(x) - \mathrm{x}(0)$$

to obtain

$$\begin{array}{rcl} sL(\mathrm{x}) - \mathrm{x}(0) & = & L(C\mathrm{x}) \\ sL(\mathrm{x}) - L(C\mathrm{x}) & = & \mathrm{x}(0) \\ sI\,L(\mathrm{x}) - C\,L(\mathrm{x}) & = & \mathrm{x}(0) \\ (sI - C)L(\mathrm{x}) & = & \mathrm{x}(0). \end{array}$$

Resolvent

The inverse of $sI - C$ is called the **resolvent**, a term invented to describe the equation

$$L(\mathrm{x}(t)) = (sI - C)^{-1}\mathrm{x}(0).$$

An Illustration for $\mathbf{x}' = C\mathbf{x}$

Define $C = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives a scalar initial value problem

$$\begin{cases} x_1'(t) &= 2x_1(t) &+ &3x_2(t), \\ x_2'(t) &= & &4x_2(t), \\ x_1(0) &= 1, \\ x_2(0) &= 2. \end{cases}$$

Then the adjugate formula $A^{-1} = \dfrac{\mathsf{adj}(A)}{\det(A)}$ gives the resolvent

$$(sI - C)^{-1} = \frac{1}{(s-2)(s-4)} \begin{pmatrix} s-4 & 3 \\ 0 & s-2 \end{pmatrix}.$$

The Laplace transform of the solution is then

$$\boldsymbol{L}(\mathrm{x}(\boldsymbol{t})) = (\boldsymbol{sI} - \boldsymbol{C})^{-1} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{s}+\mathbf{2}}{(\boldsymbol{s}-\mathbf{2})(\boldsymbol{s}-\mathbf{4})} \\ \dfrac{\mathbf{2}}{\boldsymbol{s}-\mathbf{4}} \end{pmatrix}.$$

Partial fractions and use of the backward Laplace table imply

$$\mathrm{x}(\boldsymbol{t}) = \begin{pmatrix} \mathbf{3}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} - \mathbf{2}\boldsymbol{e}^{\mathbf{2}\boldsymbol{t}} \\ \mathbf{2}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} \end{pmatrix}.$$