An RREF Method for Finding Inverses

An efficient method to find the inverse $\boldsymbol{B}$ of a square matrix $\boldsymbol{A}$, should it happen to exist, is to form the augmented matrix $\boldsymbol{C} = \mathbf{aug}(\boldsymbol{A}, \boldsymbol{I})$ and then read off $\boldsymbol{B}$ as the package of the last $\boldsymbol{n}$ columns of $\mathbf{rref}(\boldsymbol{C})$. This method is based upon the equivalence

$$\mathbf{rref}(\mathbf{aug}(\boldsymbol{A}, \boldsymbol{I})) = \mathbf{aug}(\boldsymbol{I}, \boldsymbol{B}) \quad \text{if and only if} \quad \boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}.$$

Main Results

Theorem 1 (Inverse Test)
If A and B are square matrices such that $AB = I$, then also $BA = I$. Therefore, only one of the equalities $AB = I$ or $BA = I$ is required to check an inverse.

Theorem 2 (The rref Inversion Method)
Let A and B denote square matrices. Then

(a) If $\mathrm{rref}(\mathrm{aug}(A, I)) = \mathrm{aug}(I, B)$, then $AB = BA = I$ and B is the inverse of A.

(b) If $AB = BA = I$, then $\mathrm{rref}(\mathrm{aug}(A, I)) = \mathrm{aug}(I, B)$.

(c) If $\mathrm{rref}(\mathrm{aug}(A, I)) = \mathrm{aug}(C, B)$ and $C \neq I$, then A is not invertible.

Finding inverses

The $\mathbf{rref}$ inversion method will be illustrated for the matrix

$$\boldsymbol{C} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

Define the first frame of the sequence to be $\boldsymbol{C}_1 = \mathbf{aug}(\boldsymbol{C}, \boldsymbol{I})$, then compute the frame sequence to $\mathbf{rref}(\boldsymbol{C})$ as follows.

$$C_1 = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

First Frame

$$C_2 = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right)$$

`combo(3,2,-1)`

$$C_3 = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

`mult(3,1/2)`

$$C_4 = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

`combo(3,2,1)`

`combo(3,1,-1)`

$$C_5 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

Last Frame

Extract the Inverse Matrix

The theory

$$\mathbf{rref}(\mathbf{aug}(\boldsymbol{A},\boldsymbol{I})) = \mathbf{aug}(\boldsymbol{I},\boldsymbol{B}) \quad \text{if and only if} \quad \boldsymbol{AB} = \boldsymbol{I}$$

implies that the inverse of $\boldsymbol{A}$ is the matrix in the right panel of the last frame

$$\boldsymbol{C}_5 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right).$$

Then

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix}.$$