Chapter 10

Phase Plane Methods

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Studied here are planar autonomous systems of differential equations. The topics:

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10.1 Autonomous Planar Systems

A set of two scalar differential equations of the form

$$(1) \qquad \begin{aligned} x'(t) &= F(x(t), y(t)), \\ y'(t) &= G(x(t), y(t)). \end{aligned}$$

is called a **planar autonomous system**. The term **autonomous** means **self-governing**, justified by the absence of the time variable t in the functions $F(x, y)$, $G(x, y)$.

To obtain the vector form, let $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\mathbf{f}(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$ and write (1) as the first order vector-matrix system

$$(2) \qquad \mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)).$$

It is assumed that F, G are continuously differentiable in some region $\mathcal{D}$ in the xy-plane. This assumption makes $\mathbf{f}$ continuously differentiable in $\mathcal{D}$ and guarantees that Picard's existence-uniqueness theorem for initial value problems applies to the initial value problem $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$, $\mathbf{x}(0) = \mathbf{x}_0$. Accordingly, to each $\mathbf{x}_0 = (x_0, y_0)$ in $\mathcal{D}$ there corresponds a unique solution $\mathbf{x}(t) = (x(t), y(t))$, represented as a planar curve in the xy-plane, which passes through $\mathbf{x}_0$ at $t = 0$.

Such a planar curve is called a **trajectory** of the system and its parameter interval is some maximal interval of existence $T_1 < t < T_2$, where T_1 and T_2 might be infinite. The graphic of a trajectory drawn as a parametric curve in the xy-plane is called a **phase portrait** and the xy-plane in which it is drawn is called the **phase plane**.

Trajectories don't cross. Autonomy of the planar system plus uniqueness of initial value problems implies that trajectories $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ cannot touch or cross. Hand-drawn phase portraits are accordingly limited: *you cannot draw a solution trajectory that touches another solution curve*!

Theorem 1 (Identical trajectories)
Assume that Picard's existence-uniqueness theorem applies to initial value problems in $\mathcal{D}$ for the planar system (1). Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two trajectories of system (1). If times t_1, t_2 exist such that

$$(3) \qquad x_1(t_1) = x_2(t_2), \quad y_1(t_1) = y_2(t_2),$$

then for the value $c = t_1 - t_2$ the equations $x_1(t+c) = x_2(t)$ and $y_1(t+c) = y_2(t)$ are valid for all allowed values of t. This means that the two trajectories are on one and the same planar curve, or in the contrapositive, two different trajectories cannot touch or cross in the phase plane.

Proof: Define $x(t) = x_1(t+c)$, $y(t) = y_1(t+c)$. By the chain rule, $(x(t), y(t))$ is a solution of the planar system, because $x'(t) = x_1'(t+c) = F(x_1(t+c), y_1(t+c)) = F(x(t), y(t))$, and similarly for the second differential equation. Further, (3) implies $x(t_2) = x_2(t_2)$ and $y(t_2) = y_2(t_2)$, therefore Picard's uniqueness theorem implies that $x(t) = x_2(t)$ and $y(t) = y_2(t)$ for all allowed values of t. The proof is complete.

Equilibria. A trajectory that reduces to a point, or a constant solution $x(t) = x_0$, $y(t) = y_0$, is called an **equilibrium solution**. The equilibrium solutions or **equilibria** are found by solving the nonlinear equations

$$F(x_0, y_0) = 0, \quad G(x_0, y_0) = 0.$$

Each such (x_0, y_0) in $\mathcal{D}$ is a trajectory whose graphic in the phase plane is a single point, called an **equilibrium point**. In applied literature, it may be called a **critical point**, **stationary point** or **rest point**. Theorem 1 has the following geometrical interpretation.

> Assuming uniqueness, no other trajectory $(x(t), y(t))$ in the phase plane can touch an equilibrium point (x_0, y_0).

Equilibria (x_0, y_0) are often found from linear equations

$$ax_0 + by_0 = e, \quad cx_0 + dy_0 = f,$$

which are solved by linear algebra methods. They constitute an important subclass of algebraic equations which can be solved symbolically. In this special case, symbolic solutions exist for the equilibria.

It is interesting to report that in a practical sense the equilibria may be reported incorrectly, due to the limitations of computer software, even in this case when exact symbolic solutions are available. An example is $x' = x + y$, $y' = \epsilon y - \epsilon$ for small $\epsilon > 0$. The root of the problem is translation of ϵ to a machine constant, which is zero for small enough ϵ. The result is that computer software detects infinitely many equilibria when in fact there is exactly one equilibrium point. This example suggests that symbolic computation be used by default.

Practical methods for computing equilibria. There is no supporting theory to find equilibria for all choices of F and G. However, there is a rich library of special methods for solving nonlinear algebraic equations, including celebrated numerical methods such as **Newton's method** and the **bisection method**. Computer algebra systems like `maple` and `mathematica` offer convenient codes to solve the equations, when possible, including symbolic solutions. Applied mathematics relies heavily on the dynamically expanding library of special methods, which grows monthly due to new mathematical discoveries.

Population biology. Planar autonomous systems have been applied to two-species populations like two species of trout, who compete for food from the same supply, and foxes and rabbits, who compete in a predator-prey situation.

Trout system. Certain equilibria are significant, because they represent the population sizes for **cohabitation**. A point in the phase space that is not an equilibrium point corresponds to population sizes that cannot coexist, they must change with time. Some equilibria are consequently **observable** or **average** population sizes while non-equilibria correspond to snapshot population sizes that are subject to flux. Biologists expect population sizes of such two-species competition models to undergo change until they reach approximately the observable values.

Rabbit-fox system. This is an example of a **predator-prey** system, in which the expected observable population sizes oscillate periodically over time. Certain equilibria for these systems represent **ideal cohabitation**. Biological experiments suggest that initial population sizes close to the equilibrium values cause populations to stay near the initial sizes, even though the populations oscillate periodically. Observations by biologists of large population variations seem to verify that individual populations oscillate periodically around the ideal cohabitation sizes.

Trout system. Consider a population of two species of trout who compete for the same food supply. A typical autonomous planar system for the species x and y is

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Equilibria. The equilibrium solutions for this system are

$$(0,0), \quad (90,0), \quad (0,60), \quad (80,20).$$

Only nonnegative population sizes are physically significant. Units for the population sizes might be in hundreds or thousands of fish. The equilibrium $(0,0)$ corresponds to **extinction** of both species, while $(0,60)$ and $(90,0)$ correspond to the unusual situation of extinction for one species. The last equilibrium $(80,20)$ corresponds to **co-existence** of the two trout species with observable population sizes of 80 and 20.

Phase Portraits

A graphic which contains all the equilibria and typical trajectories or **orbits** of a planar autonomous system (1) is called a **phase portrait**.

While graphing equilibria is not a challenge, graphing typical trajectories seems to imply that we are going to solve the differential system. This is not the case. The plan is this:

Equilibria	Plot in the xy-plane all equilibria of (1).
Window	Select an x-range and a y-range for the graph window which includes all significant equilibria (Figure 3).
Grid	Plot a uniform grid of N grid points ($N \approx 50$ for hand work) within the graph window, to populate the graphical white space (Figure 4). The isocline method might also be used to select grid points.
Field	Draw at each grid point a short tangent vector, a **replacement curve** for a solution curve through a grid point on a small time interval (Figure 5).
Orbits	Draw additional threaded trajectories on long time intervals into the remaining white space of the graphic (Figure 6). This is guesswork, based upon tangents to threaded trajectories matching nearby field tangents drawn in the previous step. See Figure 1 for matching details.

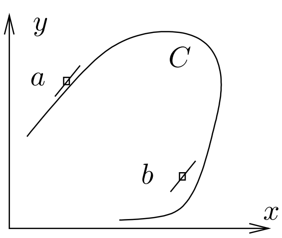

Figure 1. Badly threaded orbit.

Threaded solution curve C correctly matches its tangent to the tangent at nearby grid point a, but it fails to match at grid point b.

Why does a tangent $\vec{T}_1$ have to match a tangent $\vec{T}_2$ at a nearby grid point (see Figure 2)? A tangent vector is given by $\vec{T} = \mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$. Hence $\vec{T}_1 = \mathbf{f}(\mathbf{u}_1)$, $\vec{T}_2 = \mathbf{f}(\mathbf{u}_2)$. However, $\mathbf{u}_1 \approx \mathbf{u}_2$ in the graphic, hence by continuity of $\mathbf{f}$ it follows that $\vec{T}_1 \approx \vec{T}_2$.

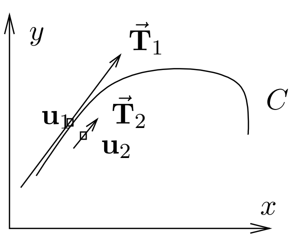

Figure 2. Tangent matching.

Threaded solution curve C matches its tangent $\vec{\mathbf{T}}_1$ at $\mathbf{u}_1$ to direction field tangent $\vec{\mathbf{T}}_2$ at nearby grid point $\mathbf{u}_2$.

It is important to emphasize that solution curves starting at a grid point are defined for a small t-interval about $t = 0$, and therefore their graphics extend on both sides of the grid point. We intend to shorten these curves until they appear to be straight line segments, graphically identical to the tangent line. Adding an arrowhead pointing in the tangent vector direction is usual. After all this construction, *the shaft of the arrow is graphically identical to a solution curve segment.* In fact, if 50 grid points were used, then 50 solution curve segments have already been entered onto the graphic! Threaded orbits are added to show what happens to solutions that are plotted on longer and longer t-intervals.

Phase portrait illustration. The method outlined above will be applied to the illustration

$$\begin{aligned} x'(t) &= x(t)+y(t), \\ y'(t) &= 1-x^2(t). \end{aligned} \tag{4}$$

The equilibria are $(1,-1)$ and $(-1,1)$. The graph window is selected as $|x| \le 2$, $|y| \le 2$, in order to include both equilibria. The uniform grid will be 11×11, although for hand work 5×5 is normal. Tangents at the grid points are short line segments which do not touch each another – they are graphically the same as short solution curves.

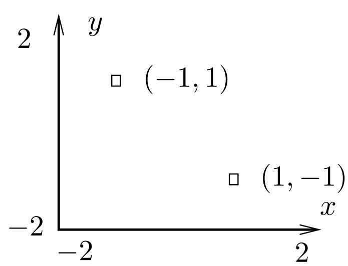

Figure 3. Equilibria $(1,-1)$**,** $(-1,1)$ **for (4) and graph window.**

The equilibria (x,y) are calculated from equations $0 = x+y$, $0 = 1-x^2$. The graph window $|x| \le 2$, $|y| \le 2$ is invented initially, then updated until Figure 5 reveals sufficiently rich field details.

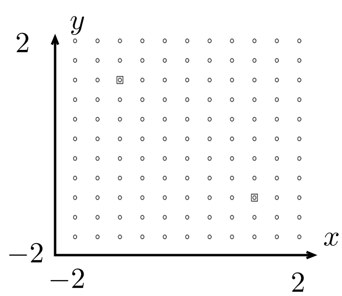

Figure 4. Equilibria $(1,-1)$**,** $(-1,1)$ **for (4) with** 11×11 **uniform grid.**

The equilibria (squares) happen to cover up two grid points (circles). The size 11×11 is invented to fill the white space in the graphic.

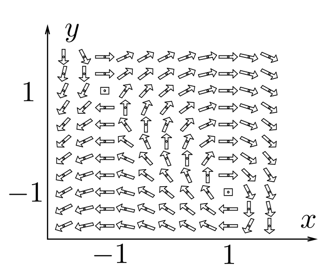

Figure 5. Equilibria for (4). Equilibria $(1,-1)$, $(-1,1)$ with an 11×11 uniform grid and direction field.

An arrow shaft at a grid point represents a solution curve over a small time interval. Threaded solution curves on long time intervals have tangents matching nearby arrow shaft directions.

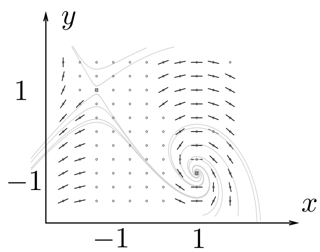

Figure 6. Equilibria for (4).
Equilibria $(1, -1)$, $(-1, 1)$ with an 11×11 uniform grid, threaded solution curves and arrow shafts from some direction field arrows.

Threaded solution curve tangents are to match nearby direction field arrow shafts. See Figure 1 for how to match tangents.

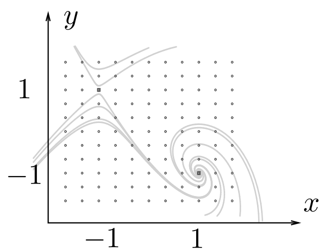

Figure 7. Phase portrait for (4).
Shown are typical solution curves and an 11×11 grid.

The direction field has been removed for clarity. Threaded solution curves do not actually cross, even though graphical resolution might suggest otherwise.

Phase plot by computer. Illustrated here is how to make the phase plot in Figure 8 with the computer algebra system `maple`.

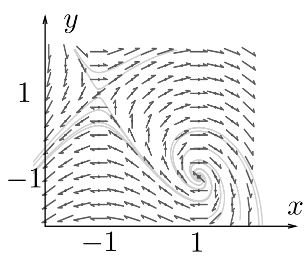

Figure 8. Phase portrait for (4).

The graphic shows typical solution curves and a direction field. Produced in `maple` using a 13×13 grid.

Before the computer work begins, the differential equation is defined and the equilibria are computed. Defaults supplied by `maple` allow an initial phase portrait to be plotted, from which the graph window is selected. The initial plot code:

```
with(DEtools):
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:
DEplot({des},[x(t),y(t)],t=-20..20,wind);
```

The initial plot suggests which initial conditions near the equilibria should be selected in order to create typical orbits on the graphic. The final code with initial data and options:

```
with(DEtools):
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:
opts:=stepsize=0.05,dirgrid=[13,13],
axes=none,thickness=3,arrows=small:
```

```
ics:=[[x(0)=-1,y(0)=1.1],[x(0)=-1,y(0)=1.5],
[x(0)=-1,y(0)=.9],[x(0)=-1,y(0)=.6],[x(0)=-1,y(0)=.3],
[x(0)=1,y(0)=-0.9],[x(0)=1,y(0)=-0.6],[x(0)=1,y(0)=-0.6],
[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=-1.6],[x(0)=1,y(0)=-1.3],
[x(0)=1,y(0)=-1.1]]:
DEplot({des},[x(t),y(t)],t=-20..20,wind,ics,opts);
```

Direction field by computer. While `maple` can produce direction fields with its `DEplot` tool, the basic code that produces a field can be written with minimal outside support, therefore it applies to other programming languages. The code below applies to the example $x' = x + y$, $y' = 1 - x^2$ treated above.

```
# 2D phase plane direction field with uniform nxm grid.
# Tangent length is 9/10 the grid box width W0.
a:=-2:b:=2:c:=-2:d:=2:n:=11:m:=11:
H:=evalf((b-a)/(n+1)):K:=evalf((d-c)/(m+1)):W0:=min(H,K):
X:=t->a+H*(t):Y:=t->c+K*(t):P:=[]:
F1:=(x,y)->evalf(x+y):F2:=(x,y)->evalf(1-x^2):
for i from 1 to n do
 for j from 1 to m do
  x:=X(i):y:=Y(j):M1:=F1(x,y): M2:=F2(x,y):
  if (M1 =0 and M2 =0) then # no tangent, make a box
   h:=W0/5:V:=plottools[rectangle]([x-h,y+h],[x+h,y-h]):
  else
   h:=evalf(((1/2)*9*W0/10)/sqrt(M1^2+M2^2)):
   p1:=x-h*M1:p2:=y-h*M2:q1:=x+h*M1:q2:=y+h*M2:
   V:=plottools[arrow]([p1,p2],[q1,q2],0.2*W0,0.5*W0,1/4):
  fi:
  if (P = []) then P:=V: else P:=P,V: fi:
od:od:
plots[display](P);
```

Maple libraries `plots` and `plottools` are used. The routine `rectangle` requires two arguments ul, lr, which are the upper left (ul) and lower right (lr) vertices of the rectangle. The routine `arrow` requires five arguments P, Q, sw, aw, af: the two points P, Q which define the arrow shaft and direction, plus the shaft width sw, arrowhead width aw and arrowhead length fraction af (fraction of the shaft length). These primitives plot a polygon from its vertices. The rectangle computes four vertices and the arrow seven vertices, which are then passed on to the `PLOT` primitive to make the graphic.

Stability

Consider an autonomous system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ with $\mathbf{f}$ continuously differentiable in a region $\mathcal{D}$ in the plane.

Stable equilibrium. An equilibrium point $\mathbf{x}_0$ in $\mathcal{D}$ is said to be **stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that

(a) given $\mathbf{x}(0)$ in $\mathcal{D}$ with $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$, then the solution $\mathbf{x}(t)$ exists on $0 \le t < \infty$ and

(b) $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$ for $0 \le t < \infty$.

Unstable equilibrium. The equilibrium point $\mathbf{x}_0$ is called **unstable** provided it is **not stable**, which means **(a)** or **(b)** fails (or both).

Asymptotically stable equilibrium. The equilibrium point $\mathbf{x}_0$ is said to be **asymptotically stable** provided **(a)** and **(b)** hold (it is **stable**), and additionally

(c) $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}_0\| = 0$ for $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$.

Applied accounts of stability tend to emphasize item **(b)**. Careful application of stability theory requires attention to **(a)**, which is the question of extension of solutions of initial value problems to the half-axis.

Basic extension theory for solutions of autonomous equations says that **(a)** will be satisfied provided **(b)** holds for those values of t for which $\mathbf{x}(t)$ is already defined. Stability verifications in mathematical and applied literature often implicitly use extension theory, in order to present details compactly. The reader is advised to adopt the same predisposition as researchers, who assume the reader to be equally clever as they.

Physical stability. In the model $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$, physical stability addresses changes in $\mathbf{f}$ as well as changes in $\mathbf{x}(0)$. The meaning is this: physical parameters of the model, e.g., the mass $m > 0$, damping constant $c > 0$ and Hooke's constant $k > 0$ in a damped spring-mass system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{c}{m}\,y - \frac{k}{m}\,x, \end{aligned}$$

may undergo small changes without significantly affecting the solution.

In physical stability, stable equilibria correspond to **physically observed** data whereas other solutions correspond to **transient observations** that disappear over time. A typical instance is the trout system

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned} \tag{5}$$

Physically observed data in the trout system (5) corresponds to the **carrying capacity**, represented by the **stable equilibrium** point $(80, 20)$, whereas transient observations are snapshot population sizes that are subject to change over time. The strange extinction equilibria $(90, 0)$ and $(0, 60)$ are **unstable equilibria**, which disagrees with intuition about zero births for less than two individuals, but agrees with graphical representations of the trout system in Figure 9. Changing f for a trout system adjusts the physical constants which describe the birth and death rates, whereas changing $\mathbf{x}(0)$ alters the initial population sizes of the two trout species.

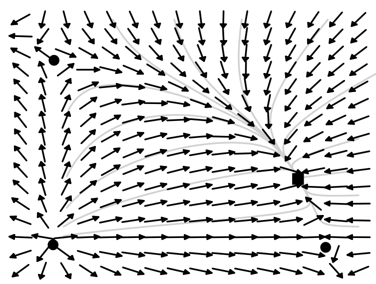

Figure 9. Phase portrait for the trout system (5).

Shown are typical solution curves and a direction field. Equilibrium $(80, 20)$ is asymptotically stable (a square). Equilibria $(0, 0)$, $(90, 0)$, $(0, 60)$ are unstable (circles).

10.2 Planar Constant Linear Systems

A **constant linear** planar system is a set of two scalar differential equations of the form

$$(1)\qquad \begin{aligned} x'(t) &= ax(t) + by(t)),\\ y'(t) &= cx(t) + dy(t)), \end{aligned}$$

where a, b, c and d are constants. In matrix form,

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Solutions drawn in phase portraits don't cross, because the system is autonomous. The origin is always an equilibrium solution. There can be infinitely many equilibria, found by solving $A\mathbf{x} = \mathbf{0}$ for the constant vector $\mathbf{x}$, when A is not invertible.

Recipe. A recipe exists for solving system (1), which parallels the recipe for second order constant coefficient equations $Ay'' + By' + Cy = 0$. The reader should view the result as an advertisement for learning Putzer's spectral method, page 618, which is used to derive the formulas.

Theorem 2 (Planar Constant Linear System Recipe)
Consider the real planar system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\mathbf{x}(t)$ is given by the formula

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0)$$

where the 2×2 real invertible matrix $\Phi(t)$ is defined as follows.

Real $\lambda_1 \ne \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} (A - \lambda_1 I).$
Real $\lambda_1 = \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + te^{\lambda_1 t} (A - \lambda_1 I).$
Complex $\lambda_1 = \overline{\lambda}_2$, $\lambda_1 = a + bi$, $b > 0$	$\Phi(t) = e^{at} \left(\cos(bt)\, I + (A - aI)\dfrac{\sin(bt)}{b} \right).$

Continuity and redundancy. The formulas are continuous in the sense that limiting $\lambda_1 \to \lambda_2$ in the first formula or $b \to 0$ in the last formula produces the middle formula for real double roots. The first formula is also valid for complex conjugate roots λ_1, $\lambda_2 = \overline{\lambda}_1$ and it reduces to the third when $\lambda_1 = a + ib$, therefore the third formula is technically redundant, but nevertheless useful, because it contains no complex numbers.

Illustrations. Typical cases are represented by the following 2×2 matrices A, which correspond to roots λ_1, λ_2 of the characteristic equation which are real distinct, real double or complex conjugate.

$\lambda_1 = 5,\ \lambda_2 = 2$ — Real distinct roots.

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \qquad \mathbf{x}(t) = \left(e^{5t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{2t}-e^{5t}}{2-5}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix} \right)\mathbf{x}(0).$$

$\lambda_1 = \lambda_2 = 3$ — Real double root.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad \mathbf{x}(t) = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}\mathbf{x}(0).$$

$\lambda_1 = \overline{\lambda}_2 = 2+3i$ — Complex conjugate roots.

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \qquad \mathbf{x}(t) = e^{2t}\begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}\mathbf{x}(0).$$

Isolated equilibria. An autonomous system is said to have an **isolated equilibrium** at $\mathbf{x} = \mathbf{x}_0$ provided $\mathbf{x}_0$ is the only constant solution of the system in $|\mathbf{x} - \mathbf{x}_0| < r$, for $r > 0$ sufficiently small.

Theorem 3 (Isolated Equilibrium)
The following are equivalent for a constant planar system $\mathbf{x}'(t) = A\mathbf{x}(t)$:

1. The system has an isolated equilibrium at $\mathbf{x} = \mathbf{0}$.

2. $\det(A) \ne 0$.

3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1\lambda_2 \ne 0$.

Proof: The expansion $\det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$ shows that $\det(A) = \lambda_1\lambda_2$. Hence $\mathbf{2} \equiv \mathbf{3}$. We prove now $\mathbf{1} \equiv \mathbf{2}$. If $\det(A) = 0$, then $A\mathbf{u} = \mathbf{0}$ has infinitely many solutions $\mathbf{u}$ on a line through $\mathbf{0}$, therefore $\mathbf{x} = \mathbf{0}$ is not an isolated equilibrium. If $\det(A) \ne 0$, then $A\mathbf{u} = \mathbf{0}$ has exactly one solution $\mathbf{u} = \mathbf{0}$, so the system has an isolated equilibrium at $\mathbf{x} = \mathbf{0}$.

Classification of isolated equilibria. For linear equations

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

we explain the phase portrait classifications **saddle, node, spiral, center** near the isolated equilibrium point $\mathbf{x} = \mathbf{0}$, and how to detect them when they occur. Below, λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

The reader is directed to Figures 10–14 for illustrations of the classifications.

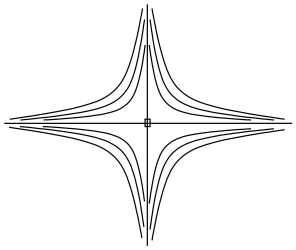

Figure 10. Saddle

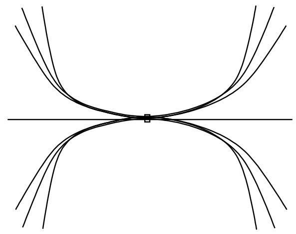

Figure 11. Improper node

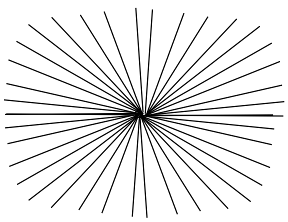

Figure 12. Proper node

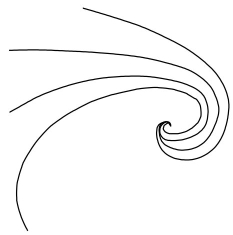

Figure 13. Spiral

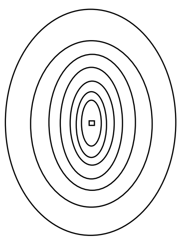

Figure 14. Center

Saddle λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$

A **saddle** has solution formula

$$\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{c}_1 + e^{\lambda_2 t}\mathbf{c}_2,$$
$$\mathbf{c}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\mathbf{x}(0), \quad \mathbf{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\mathbf{x}(0).$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

Case $\lambda_1 = \lambda_2$. An **improper node** has solution formula

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{c}_1 + t\mathbf{c}_2\right),$$
$$\mathbf{c}_1 = \mathbf{x}(0), \quad \mathbf{c}_2 = (A - \lambda_1 I)\mathbf{x}(0).$$

An improper node is further classified as a **degenerate node** ($\mathbf{c}_2 \neq \mathbf{0}$) or a **star node** ($\mathbf{c}_2 = \mathbf{0}$).

Assume $\lambda_1 = \lambda_2 < 0$. A **degenerate node** phase portrait has all trajectories tangent at $t = \infty$ to direction $\mathbf{c}_2$. A **star node** phase portrait consists of trajectories $\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{c}_1$, a straight line, with limit $\mathbf{0}$ at $t = \infty$. Vector $\mathbf{c}_1$ can be any direction. If $\lambda_1 > 0$, then the same is true with $t = \infty$ replaced by $t = -\infty$.

Case $\lambda_1 \neq \lambda_2$. A **proper node** is any node that is not improper. Its solution formula is

$$\begin{aligned}
&\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{c}_1 + e^{\lambda_2 t}\mathbf{c}_2, \\
&\mathbf{c}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\mathbf{x}(0), \quad \mathbf{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\mathbf{x}(0).
\end{aligned}$$

A trajectory near a proper node satisfies, for some direction $\mathbf{v}$, $\lim_{t\to\omega} \mathbf{x}'(t)/|\mathbf{x}'(t)| = \mathbf{v}$, for either $\omega = \infty$ or $\omega = -\infty$. Briefly, $\mathbf{x}(t)$ is tangent to $\mathbf{v}$ at $t = \omega$. Further, to each direction $\mathbf{v}$ corresponds some $\mathbf{x}(t)$ tangent to $\mathbf{v}$.

Spiral $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a \neq 0$, $b > 0$.

A **spiral** has solution formula

$$\begin{aligned}
&\mathbf{x}(t) = e^{at}\cos(bt)\,\mathbf{c}_1 + e^{at}\sin(bt)\,\mathbf{c}_2, \\
&\mathbf{c}_1 = \mathbf{x}(0), \quad \mathbf{c}_2 = \frac{A - aI}{b}\,\mathbf{x}(0).
\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b times an exponential amplitude which grows if $a > 0$ and decays if $a < 0$. An orbit in the phase plane **spirals out** if $a > 0$ and **spirals in** if $a < 0$.

Center $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a = 0$, $b > 0$

A **center** has solution formula

$$\begin{aligned}
&\mathbf{x}(t) = \cos(bt)\,\mathbf{c}_1 + \sin(bt)\,\mathbf{c}_2, \\
&\mathbf{c}_1 = \mathbf{x}(0), \quad \mathbf{c}_2 = \frac{1}{b}\,A\mathbf{x}(0).
\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b. Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Attractor and repeller. An equilibrium point is called an **attractor** provided solutions starting nearby limit to the point as $t \to \infty$. A **repeller** is an equilibrium point such that solutions starting nearby limit to the point as $t \to -\infty$. Terms like **attracting node** and **repelling spiral** are defined analogously.

Almost linear systems

A nonlinear planar autonomous system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ is called **almost linear** at equilibrium point $\mathbf{x} = \mathbf{x}_0$ if

$$\mathbf{f}(\mathbf{x}) = A(\mathbf{x} - \mathbf{x}_0) + \mathbf{g}(\mathbf{x}),$$
$$\lim_{\|\mathbf{x}-\mathbf{x}_0\|\to 0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

The function $\mathbf{g}$ has the same smoothness as $\mathbf{f}$. We investigate the possibility that a local phase diagram at $\mathbf{x} = \mathbf{x}_0$ for the nonlinear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ is graphically identical to the one for the linear system $\mathbf{y}'(t) = A\mathbf{y}(t)$ at $\mathbf{y} = 0$.

The results will apply to **all isolated equilibria** of $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$. This is accomplished by expanding f in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of A and g, depending upon which equilibrium point $\mathbf{x}_0$ was considered.

Define the **Jacobian matrix** of $\mathbf{f}$ at equilibrium point $\mathbf{x}_0$ by the formula

$$J = \mathbf{aug}\left(\partial_1 \mathbf{f}(\mathbf{x}_0), \partial_2 \mathbf{f}(\mathbf{x}_0)\right).$$

Taylor's theorem for functions of two variables says that

$$\mathbf{f}(\mathbf{x}) = J(\mathbf{x} - \mathbf{x}_0) + \mathbf{g}(\mathbf{x})$$

where $\mathbf{g}(\mathbf{x})/\|\mathbf{x}-\mathbf{x}_0\| \to 0$ as $\|\mathbf{x}-\mathbf{x}_0\| \to 0$. Therefore, for $\mathbf{f}$ continuously differentiable, we may always take $A = J$ to obtain from the almost linear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ its **linearization** $y'(t) = A\mathbf{y}(t)$.

Phase diagrams. For planar almost linear systems $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$, phase diagrams have been studied extensively, by Poincaré-Bendixson and a long list of researchers. It is known that only a finite number of local phase diagrams are possible near each isolated equilibrium point of the nonlinear system, the library of figures being identical to those possibilities for a linear system $\mathbf{y}'(t) = A\mathbf{y}(t)$. A precise statement, without proof, appears below.

Theorem 4 (Phase diagrams of almost linear systems)
Let the planar almost linear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ be given with $\mathbf{f}(\mathbf{x}) = A(\mathbf{x} - \mathbf{x}_0) + \mathbf{g}(\mathbf{x})$ near the isolated equilibrium point $\mathbf{x}_0$ (an isolated root of $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$). Let λ_1, λ_2 be the roots of $\det(A - \lambda I) = 0$. Then:

1. If $\lambda_1 = \lambda_2$, then the equilibrium $\mathbf{x}_0$ of the nonlinear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ is either a node or a spiral. The equilibrium $\mathbf{x}_0$ is an asymptotically stable attractor if $\lambda_1 < 0$ and it is a repeller if $\lambda_1 > 0$.

2. If $\lambda_1 = \overline{\lambda}_2 = ib$, $b > 0$, then the equilibrium $\mathbf{x}_0$ of the nonlinear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ is either a center or a spiral. The stability of the equilibrium $\mathbf{x}_0$ cannot be predicted from properties of A.

3. In all other cases, the isolated equilibrium $\mathbf{x}_0$ has graphically the same phase diagram as the associated linear system $\mathbf{y}'(t) = A\mathbf{y}(t)$ at $\mathbf{y} = \mathbf{0}$. In particular, local phase diagrams of a saddle, spiral or node can be graphed from the linear system. The local properties of stability, instability and asymptotic stability at $\mathbf{x}_0$ are inherited by the nonlinear system from the linear system.

Classification of equilibria. A system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$ has a local phase diagram determined by the linear system $\mathbf{y}'(t) = A\mathbf{y}(t)$, except in the case when the roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ are equal or purely imaginary (see Theorem 4). To summarize:

Table 1. Equilibria classification for almost linear systems

Eigenvalues of A	Nonlinear Classification
$\lambda_1 < 0 < \lambda_2$	Unstable saddle
$\lambda_1 < \lambda_2 < 0$	Stable proper node
$\lambda_1 > \lambda_2 > 0$	Unstable proper node
$\lambda_1 = \lambda_2 < 0$	Stable improper node or spiral
$\lambda_1 = \lambda_2 > 0$	Unstable improper node or spiral
$\lambda_1 = \overline{\lambda}_2 = a + ib$, $a < 0$, $b > 0$	Stable spiral
$\lambda_1 = \overline{\lambda}_2 = a + ib$, $a > 0$, $b > 0$	Unstable spiral
$\lambda_1 = \overline{\lambda}_2 = ib$, $b > 0$	Stable or unstable, center or spiral

Nonlinear classifications of equilibria. Applied literature may refer to an equilibrium point $\mathbf{x}_0$ of a nonlinear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ as a saddle, node, center or spiral. The geometry of these classifications is explained below.

Saddle. The term implies that *locally* the phase diagram looks like a linear saddle. In nonlinear phase diagrams, the straight lines to which orbits are asymptotic appear to be curves instead. These curves are called **separatrices**, and they generally are unions of certain orbits and equilibria.

Node. Each orbit starting near the equilibrium is expected to limit to the equilibrium at either $t = \infty$ (stable) or $t = -\infty$ (unstable), in a fashion asymptotic to a direction $\mathbf{v}$. The terminology generally applies to linearized system star points, in which case there is an orbit asymptotic to $\mathbf{v}$ for every direction $\mathbf{v}$. If there is only one direction $\mathbf{v}$ possible, or all orbits are asymptotic to just one separatrix, then the equilibrium is still classified as a node.

Center. Locally, orbits are periodic solutions which enclose the equilibrium point. The periodic orbit plus its interior limits as a planar region to the equilibrium point. Drawings often portray the periodic orbit as a convex figure, but this is not correct, in general, because the periodic orbit can have any shape. In particular, the linearized system may have phase diagram consisting of concentric circles, but the nonlinear phase diagram has no such exact geometric structure.

Spiral. To describe a **nonlinear spiral**, we require that an orbit starting on a given ray emanating from the equilibrium point must intersect that ray in infinitely many distinct points on $(-\infty, \infty)$. The intuitive notion of a **nonlinear spiral** is obtained from a linear example, e.g.,

$$\mathbf{u}'(t) = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{u}(t).$$

The component solutions $x(t) = e^{-t}(A \cos 2t + B \sin 2t)$, $y(t) = e^{-t}(-A \sin 2t + B \cos 2t)$ oscillate infinity often on $(-\infty, \infty)$ and the orbit rotates around the equilibrium point with non-constant time-varying amplitude.

10.3 Biological Models

Studied here are **predator-prey models** and **competition models** for two populations. Assumed as background from population biology are the one-dimensional Malthusian model $P' = kP$ and the one-dimensional Verhulst model $P' = (a - bP)P$.

Predator-Prey Models

One species called the **predator** feeds on the other species called the **prey**. The prey feeds on some constantly available food supply, e.g., rabbits eat plants and foxes eat rabbits.

Credited with the classical predator-prey model is the Italian mathematician **Vito Volterra** (1860-1940), who worked on cyclic variations in shark and prey-fish populations in the Adriatic sea. The following biological assumptions apply to model a predator-prey system.

Malthusian growth	The prey population grows according to the growth equation $x'(t) = ax(t)$, $a > 0$, in the absence of predators.
Malthusian decay	The predator population decays according to the decay equation $y'(t) = -by(t)$, $b > 0$, in the absence of prey.
Chance encounters	The prey decrease population at a rate $-pxy$, $p > 0$, due to chance encounters with predators. The predators increase population due to the chance interactions at a rate qxy, $q > 0$.

The interaction terms qxy and $-pxy$ are justified by arguing that the frequency of chance encounters is proportional to the product xy. Biologists explain the proportionality by saying that doubling either population should double the frequency of chance encounters. Adding the Malthusian rates and the chance encounter rates gives the **Volterra predator-prey system**

$$
\begin{array}{lrcl}
(1) & x'(t) & = & (a - py(t))x(t), \\
 & y'(t) & = & (qx(t) - b)y(t).
\end{array}
$$

The differential equations are displayed in this form in order to emphasize that each of $x(t)$ and $y(t)$ satisfy a scalar first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time. For initial population sizes near zero, the two differential equations behave very much like the Malthusian growth model $x'(t) = ax(t)$ and the Malthusian decay model $y'(t) = -by(t)$. This basic growth/decay property allows us to identify the predator variable y (or the prey variable

x), regardless of the order in which the differential equations are written. As viewed from Malthus' law $u' = ru$, the prey population has growth rate $r = a - py$ which gets smaller as the number y of predators grows, resulting in fewer prey. Likewise, the predator population has decay rate $r = -b + qx$, which gets larger as the number x of prey grows, causing increased predation. These are the basic ideas of Verhulst, applied to the individual populations x and y.

Equilibria. The equilibrium points $\mathbf{x}$ satisfy $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ where $\mathbf{f}$ is defined by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} (a-py)x \\ (qx-b)y) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

The equilibria are found to be $(0,0)$ and $(b/q, a/p)$.

Linearized predator-prey system. The function $\mathbf{f}$ defined by (2) has vector partial derivatives

$$\partial_x \mathbf{f}(\mathbf{x}) = \begin{pmatrix} a - py \\ qy \end{pmatrix}, \quad \partial_y \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -px \\ qx - b \end{pmatrix}.$$

The Jacobian matrix $J = \begin{pmatrix} \partial_x \mathbf{f}(\mathbf{x}) & \partial_y \mathbf{f}(\mathbf{x}) \end{pmatrix}$ is given explicitly by

$$J = \begin{pmatrix} a - py & -px \\ qy & qx - b \end{pmatrix}. \tag{3}$$

The matrix J is evaluated at an equilibrium point (a root of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$) to obtain a 2×2 matrix A for the linearized system $\mathbf{x}'(t) = A\mathbf{x}(t)$. The linearized systems are:

Equilibrium $(0,0)$ $\qquad \mathbf{x}'(t) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \mathbf{x}(t)$

Equilibrium $(b/q, a/p)$ $\qquad \mathbf{x}'(t) = \begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix} \mathbf{x}(t)$

The first equilibrium $(0,0)$ is classified as a **saddle**, therefore the almost linear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ has a saddle at the origin.

The second equilibrium $(b/q, a/p)$ is classified as a **center**, therefore the almost linear system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$ is either a center or a spiral. We shall eliminate the spiral case by applying the following result, whose proof is outlined in the exercises.

Theorem 5 (Predator-prey general solution)

Let $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be an orbit of the predator-prey system (1) with

$x(0) > 0$ and $y(0) > 0$. Then for some constant C,

$$a \ln y(t) + b \ln x(t) - qx(t) - py(t) = C. \tag{4}$$

Proof: Assume the equilibrium $(b/q, a/p)$ is *not a center*, then it is a spiral point and some orbit touches the line $x = b/q$ in points $(b/q, u_1)$, $(b/q, u_2)$ with $u_1 \neq u_2$, $u_1 > a/p$, $u_2 > a/p$. Consider the energy function $E(u) = a \ln u - pu$. Due to relation (4), $E(u_1) = E(u_2) = E_0$, where $E_0 \equiv C + b - b\ln(b/q)$. By the Mean Value Theorem of calculus, $(u_1 - u_2)dE/du = 0$ at some u between u_1 and u_2. This is a contradiction, because $dE/du = (a - pu)/u$ is strictly negative for $a/p < u < \infty$. Therefore, the equilibrium $(b/q, a/p)$ is center.

Rabbits and foxes. A typical application of predator-prey theory is the Volterra population model for x rabbits and y foxes given by the system of differential equations

$$\begin{aligned} x'(t) &= 0.004\, x(t)(40 - y(t)), \\ y'(t) &= 0.02\, y(t)(x(t) - 60). \end{aligned} \tag{5}$$

The equilibria of system (5) are $(0, 0)$ and $(60, 40)$. A phase plot for system (5) appears in Figure 15.

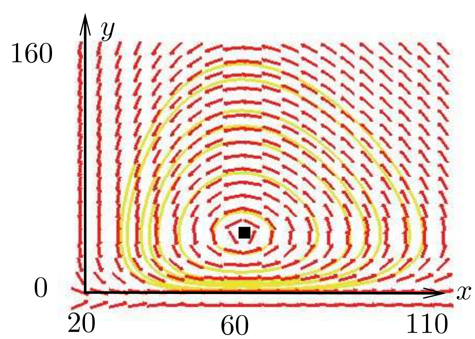

Figure 15. Oscillating population of rabbits and foxes (5).

Equilibria $(0, 0)$ and $(60, 40)$ are respectively a saddle and a center. The period of oscillation is about 17 for the largest orbit and about 14.5 for the smallest orbit.

The linearized system at $(60, 40)$ is

$$\begin{aligned} x'(t) &= -\frac{6}{25} y(t), \\ y'(t) &= \frac{4}{5} x(t). \end{aligned}$$

This system implies $x''(t) + \frac{6}{25}\frac{4}{5}x(t) = 0$, a harmonic oscillator of period $2\pi/\sqrt{24/125} \approx 14.33934302$. Therefore, the period of smaller and smaller orbits enclosing the equilibrium $(60, 40)$ must approach a value that is approximately 14.339.

The **fluctuations** in population size $x(t)$ are measured graphically by the maximum and minimum values of x in the phase diagram, or more simply, by graphing t versus $x(t)$ in a planar graphic. To illustrate, the orbit for $x(0) = 60$, $y(0) = 100$ is graphed in Figure 16, from which it is determined that the rabbit population $x(t)$ fluctuates between 39 and 87. Similar remarks apply to foxes $y(t)$.

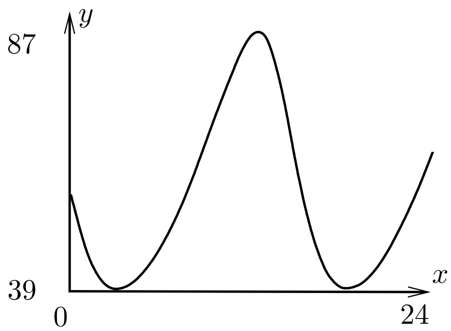

Figure 16. Plot of time t versus rabbits $x(t)$ in (5) for $x(0) = 60$, $y(0) = 100$.

An initial rabbit population of 60 and fox population of 100 causes the rabbit population to fluctuate from 39 to 87.

Competition Models

Two populations **1** and **2** feed on some constantly available food supply, e.g., two kinds of insects feed on fallen fruit.

The following biological assumptions apply to model a two-population competition system.

Verhulst model **1**	Population **1** grows or decays according to the logistic equation $x'(t) = (a - bx(t))x(t)$, in the absence of population **2**.
Verhulst model **2**	Population **2** grows or decays according to the logistic equation $y'(t) = (c - dy(t))y(t)$, in the absence of population **1**.
Chance encounters	Population **1** decays at a rate $-pxy$, $p > 0$, due to chance encounters with population **2**. Population **2** decays at a rate $-qxy$, $q > 0$, due to chance encounters with population **1**.

Adding the Verhulst rates and the chance encounter rates gives the **Volterra competition system**

$$
\begin{aligned}
x'(t) &= (a - bx(t) - py(t))x(t), \\
y'(t) &= (c - dy(t) - qx(t))y(t).
\end{aligned} \tag{6}
$$

The equations show that each population satisfies a time-varying first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time. For initial population sizes near zero, the two differential equations essentially reduce to the Malthusian growth models $x'(t) = ax(t)$ and $y'(t) = cy(t)$. As viewed from Malthus' law $u' = ru$, population **1** has growth rate $r = a - bx - py$ which decreases if population **2** grows, resulting in a reduction of population **1**. Likewise, population **2** has growth rate $r = c - dy - qx$, which reduces population **2** as population

1 grows. While a, c are Malthusian growth rates, constants b, d measure **inhibition** (due to lack of food or space) and constants p, q measure **competition**.

Equilibria. The equilibrium points $\mathbf{x}$ satisfy $f(\mathbf{x}) = \mathbf{0}$ where f is defined by

$$(7) \qquad f(\mathbf{x}) = \begin{pmatrix} (a - bx - py)x \\ (c - dy - qx)y \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To isolate the most important applications, the assumption will be made of exactly four roots in population quadrant I. This is equivalent to the condition $bd - qp \neq 0$ plus all equilibria have nonnegative coordinates.

Three of the four equilibria are found to be $(0, 0)$, $(a/b, 0)$, $(0, c/d)$. The last two represent the carrying capacities of the Verhulst models in the absence of the second population. The fourth equilibrium (x_0, y_0) is found as the *unique root* of the linear system

$$\begin{pmatrix} b & p \\ q & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

which according to Cramer's rule is

$$x_0 = \frac{ad - pc}{bd - qp}, \quad y_0 = \frac{bc - qa}{bd - qp}.$$

Linearized competition system. The function f defined by (7) has vector partial derivatives

$$\partial_x f(\mathbf{x}) = \begin{pmatrix} a - 2bx - py \\ -qy \end{pmatrix}, \quad \partial_y f(\mathbf{x}) = \begin{pmatrix} -px \\ c - 2dy - qx \end{pmatrix}.$$

The Jacobian matrix $J = \begin{pmatrix} \partial_x f(\mathbf{x}) & \partial_y f(\mathbf{x}) \end{pmatrix}$ is given explicitly by

$$(8) \qquad J(x, y) = \begin{pmatrix} a - 2bx - py & -px \\ -qy & c - 2dy - qx \end{pmatrix}.$$

The matrix J is evaluated at an equilibrium point (a root of $f(\mathbf{x}) = \mathbf{0}$) to obtain a 2×2 matrix A for the linearized system $\mathbf{x}'(t) = A\mathbf{x}(t)$. The linearized systems are:

Equilibrium $(0, 0)$
Unstable node or spiral

$$\mathbf{x}'(t) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mathbf{x}(t)$$

Equilibrium $(a/b, 0)$
Saddle or nodal sink

$$\mathbf{x}'(t) = \begin{pmatrix} -a & -ap/b \\ 0 & c - qa/b \end{pmatrix} \mathbf{x}(t)$$

Equilibrium $(0, c/d)$ Saddle or nodal sink	$\mathbf{x}'(t) = \begin{pmatrix} a - cp/d & 0 \\ -qc/d & -c \end{pmatrix} \mathbf{x}(t)$
Equilibrium (x_0, y_0) Saddle or nodal sink	$\mathbf{x}'(t) = \begin{pmatrix} -bx_0 & -px_0 \\ -qy_0 & -dy_0 \end{pmatrix} \mathbf{x}(t)$

Equilibria $(a/b, 0)$ and $(0, c/d)$ are either both saddles or both nodal sinks, accordingly as $bd - qp > 0$ or $bd - qp < 0$, because of the requirement that a, b, c, d, p, q, x_0, y_0 be positive.

The analysis of equilibrium (x_0, y_0) is made by computing the eigenvalues of the linearized system to be

$$\frac{1}{2}\left(-(bx_0 + dy_0) \pm \sqrt{D}\right), \quad D = (bx_0 - dy_0)^2 + 4pqx_0y_0.$$

Because $D > 0$, the equilibrium is a saddle when the roots have opposite sign, and it is a nodal sink when both roots are negative. The saddle case is $D > (bx_0 + dy_0)^2$ or equivalently $4x_0y_0(pq - bd) > 0$, which reduces to $bd - qp < 0$. In summary:

If $bd - qp > 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a saddle, saddle, nodal sink.

If $bd - qp < 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a nodal sink, nodal sink, saddle.

Biological meaning of $bd - qp$ negative or positive. The quantities bd and qp are measures of inhibition and competition.

Survival-extinction	The inequality $bd - qp < 0$ means that competition qp is large compared with inhibition bd. The equilibrium point (x_0, y_0) is unstable in this case, which biologically means that the two species cannot coexist: one species **survives** and the other becomes **extinct**.
Co-existence	The inequality $bd - qp > 0$ means that competition qp is small compared with inhibition bd. The equilibrium point (x_0, y_0) is asymptotically stable in this case, which biologically means the two species **co-exist**.

Survival of One Species

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$\begin{aligned} x'(t) &= x(t)(24 - x(t) - 2y(t)), \\ y'(t) &= y(t)(30 - y(t) - 2x(t)). \end{aligned} \tag{9}$$

We apply the general competition theory with $a = 24$, $b = 1$, $p = 2$, $c = 30$, $d = 1$, $q = 2$. The equilibrium points are $(0,0)$, $(24,0)$, $(0,30)$, $(12,6)$.

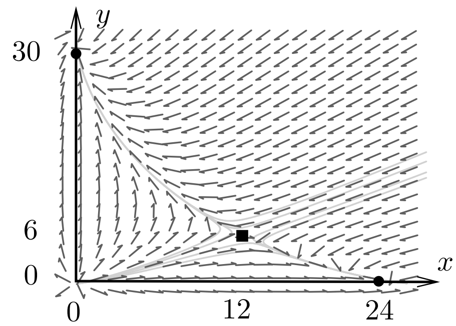

Figure 17. Survival of one species.

The equilibria are $(0,0)$, $(0,30)$, $(24,0)$ and $(12,6)$. They are classified as node, node, node, saddle, respectively. The population with initial advantage survives, while the other dies out.

Co-existence

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$(10) \qquad \begin{aligned} x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)). \end{aligned}$$

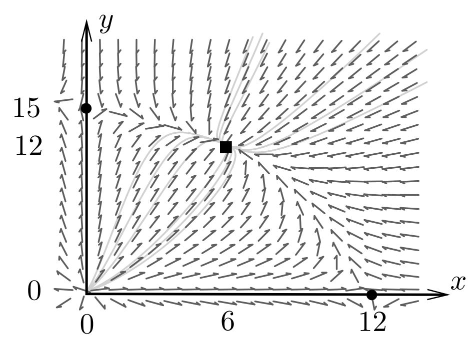

Figure 18. Coexistence.

The equilibria are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$. They are classified as node, saddle, saddle, node, respectively. A solution with $x(0) > 0$, $y(0) > 0$ limits to $(6,12)$ at $t = \infty$.

Alligators, Doomsday and Extinction

Let us assume a competition-type model (6) in which the Verhulst dynamics has doomsday-extinction type. Thus, we take the signs of a, b, c, d in (6) to be negative, but p, q are still positive. The populations $x(t)$ and $y(t)$ are unsophisticated in the sense that each population in the absence of the other is subject to only the possibilities of doomsday or extinction.

It can be verified for this general setting, although we shall not attempt to do so here, that the population quadrant $x(0) > 0$, $y(0) > 0$ is separated into two regions I and II, whose common boundary is a separatrix consisting of three equilibria and two orbits. An orbit starting in region I will have (a) $x(\infty) = 0$, $y(\infty) = \infty$, or (b) $x(\infty) = \infty$, $y(\infty) = 0$, or

(c) $x(\infty) = \infty$, $y(\infty) = \infty$. Orbits starting in region II will satisfy (d) $x(\infty) = 0$, $y(\infty) = 0$. The biological conclusion is that either population explosion (doomsday) or extinction occurs for each population.

A typical instance is:

$$\begin{aligned} x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8). \end{aligned} \tag{11}$$

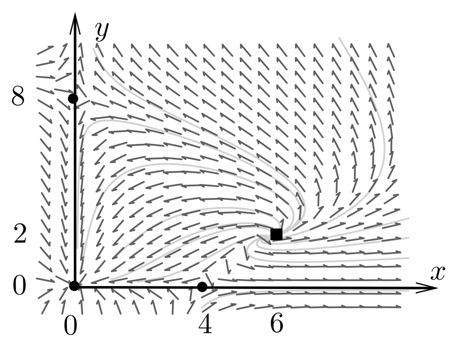

Figure 19. Phase plot of (11). Population explosion or extinction.

The equilibria are $(0,0)$, $(0,8)$, $(4,0)$ and $(6,2)$. They are classified as node, saddle, saddle, spiral, respectively. The spiral at $(6,2)$ (solid square) is an unstable source.

10.4 Mechanical Models

Nonlinear Spring-Mass System

The classical linear undamped spring-mass system is modeled by the equation $mx''(t) + kx(t) = 0$. This equation describes the excursion $x(t)$ from equilibrium $x = 0$ of a mass m attached to a spring of Hooke's constant k, with no damping and no external forces.

In the nonlinear theory, the Hooke's force term $-kx$ is replaced by a **restoring force** $F(x)$ which satisfies these four requirements:

Equilibrium 0**.** The equation $F(0) = 0$ is assumed, which gives $x = 0$ the status of a rest position.

Oddness. The equation $F(-x) = -F(x)$ is assumed, which says that the force F depends only upon the magnitude of the excursion from equilibrium, and not upon its direction. Then force F acts to **restore** the mass to its equilibrium position, like a Hooke's force $x \to kx$.

Zero damping. The damping effects always present in a real physical system are ignored. In linear approximations, it would be usual to assume a viscous damping effect $-cx'(t)$; from this viewpoint we assume $c = 0$.

Zero external force. There is no external force acting on the system. In short, only two forces act on the mass, (1) Newton's second law and (2) restoring force F.

The competition method applies to model the nonlinear spring-mass system via the two competing forces $mx''(t)$ and $F(x(t))$. The dynamical equation:

$$mx''(t) + F(x(t)) = 0. \tag{1}$$

Soft and Hard Springs

A restoring force F modeled upon Hooke's law is given by the equation $F(x) = kx$. With this force, the nonlinear spring-mass equation (1) becomes the undamped linear spring-mass system

$$mx''(t) + kx(t) = 0. \tag{2}$$

The linear equation can be thought to originate by replacing the actual spring force F by the first nonzero term of its Taylor series

$$F(x) = F(0) + F'(0)x + F''(0)\frac{x^2}{2!} + \cdots.$$

The assumptions $F(-x) = -F(x)$ and $F(0) = 0$ imply that $F(x)$ is a function of the form $F(x) = xG(x^2)$, hence all even terms in the Taylor series of F are zero.

Linear approximations to the force F drop the quadratic terms and higher from the Taylor series. More accurate nonlinear approximations are obtained by retaining extra Taylor series terms.

A restoring force F is called **hard** or **soft** provided it is given by a truncated Taylor series as follows.

Hard spring	$F(x) = kx + \beta x^3$, $\beta > 0$.
Soft spring	$F(x) = kx - \beta x^3$, $\beta > 0$.

For small excursions from equilibrium $x = 0$, a hard or soft spring force has magnitude approximately the same as the linear Hooke's force $F(x) = kx$.

Energy Conservation. Each solution $x(t)$ of the nonlinear spring-mass equation $mx''(t) + F(x(t)) = 0$ satisfies on its domain of existence the **conservation law**

$$\text{(3)} \qquad \frac{m}{2}(x'(t))^2 + \int_{x(0)}^{x(t)} F(u)\,du = C, \quad C \equiv \frac{m}{2}(x'(0))^2.$$

To prove the law, multiply the nonlinear differential equation by $x'(t)$ to obtain $mx''(t)x'(t) + F(x(t))x'(t) = 0$, then apply quadrature to obtain (3).

Kinetic and Potential Energy. Using $v = x'(t)$, the term $mv^2/2$ in (3) is called the **kinetic energy** (KE) and the term $\int_{x_0}^{x} F(u)du$ is called the **potential energy** (PE). Equation (3) says that $KE + PE = C$ or that *energy is constant* along trajectories.

The conservation laws for the soft and hard nonlinear spring-mass systems, using position-velocity notation $x = x(t)$ and $y = x'(t)$, are therefore given by the equations

$$\text{(4)} \qquad my^2 + kx^2 + \frac{1}{2}\beta x^4 = C_1, \quad C_1 = \text{constant} > 0,$$

$$\text{(5)} \qquad my^2 + kx^2 - \frac{1}{2}\beta x^4 = C_2, \quad C_2 = \text{constant}.$$

Phase Plane and Scenes. Nonlinear behavior is commonly graphed in the **phase plane**, in which $x = x(t)$ and $y = x'(t)$ are the position and velocity of the mechanical system. The plots of t versus $x(t)$ or $x'(t)$ are called **scenes**; these plots are invaluable for verifying periodic behavior and stability properties.

Hard spring

The only equilibrium for a hard spring $x' = y$, $my' = -kx - \beta x^3$ is the origin $x = y = 0$. Conservation law (4) describes a closed curve in the phase plane, which implies that trajectories are periodic orbits that encircle the equilibrium point $(0, 0)$. The classification of **center** applies. See Figures 20 and 21.

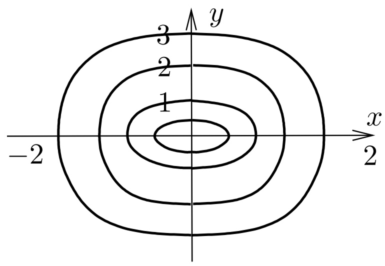

Figure 20. Hard spring $x''(t) + x(t) + 2x^3(t) = 0$**.**
Phase portrait for $x' = y$, $y' = -2x^3 - x$ on $|x| \le 2$, $|y| \le 3.5$. Initial data: $x(0) = 0$ and $y(0) = 1/2$, 1, 2, 3.

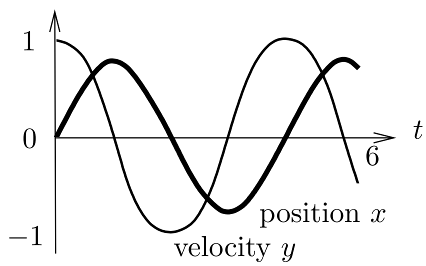

Figure 21. Hard spring $x''(t) + x(t) + 2x^3(t) = 0$**.**
Coordinate scenes for $x' = y$, $y' = -2x^3 - x$, $x(0) = 0$, $y(0) = 1$.

More intuition about the orbits can be obtained by finding the energy C_1 for each orbit. The value of C_1 decreases to zero as orbits close down upon the origin. Otherwise stated, the xyz-plot with $z = C_1$ has a minimum at the origin, which physically means that the equilibrium state $x = y = 0$ minimizes the energy. See Figure 22.

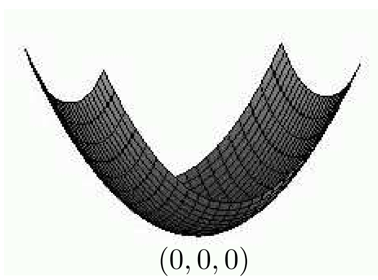

Figure 22. Hard spring energy minimization.
Plot for $x''(t) + x(t) + 2x^3(t) = 0$, using $z = y^2 + x^2 + x^4$ on $|x| \le 1/2$, $|y| \le 1$. The minimum is realized at $x = y = 0$.

Soft Spring

There are three equilibria for a soft spring

$$\begin{aligned} x' &= y, \\ my' &= -kx + \beta x^3. \end{aligned}$$

They are $(-\alpha, 0)$, $(0, 0)$, $(\alpha, 0)$, where $\alpha = \sqrt{k/\beta}$. If $(x(0), y(0))$ is given not at these points, then the mass undergoes motion. In short, the stationary mass positions are at the equilibria.

Linearization at the equilibria reveals part of the phase portrait. The linearized system at the origin is the system $x' = y$, $my' = -kx$, equivalent to the equation $mx'' + kx = 0$. It has a center at the origin. This implies the origin for the soft spring is either a center or a spiral. The other two equilibria have linearized systems equivalent to the equation $mx'' - 2kx = 0$; they are saddles.

The phase plot in Figure 23 shows separatrices, which are unions of solution curves and equilibrium points. Orbits in the phase plane, on either side of a separatrix, have physically different behavior. Shown is a center behavior interior to the union of the separatrices, while outside all orbits are unbounded.

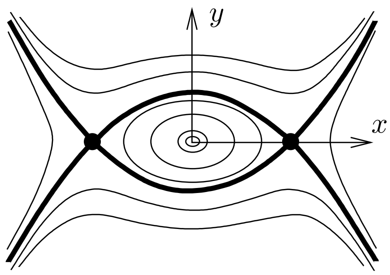

Figure 23. Soft spring
$x''(t) + x(t) - 2x^3(t) = 0$.
A phase portrait for $x' = y$, $y' = 2x^3 - x$ on $|x| \le 1.2$, $|y| \le 1.2$. The 8 separatrices are the 6 bold curves plus the two equilibria $(\sqrt{0.5}, 0)$, $(-\sqrt{0.5}, 0)$.

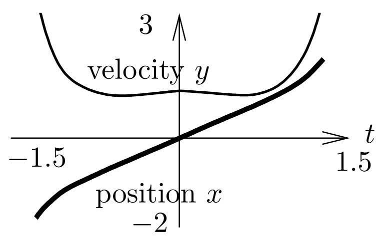

Figure 24. Soft spring
$x''(t) + x(t) - 2x^3(t) = 0$.
Coordinate scenes for $x' = y$, $y' = 2x^3 - x$, $x(0) = 0$, $y(0) = 4$.

Nonlinear Pendulum

Consider a nonlinear undamped pendulum of length L making angle $\theta(t)$ with the gravity vector. The **nonlinear pendulum equation** is given by

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0 \tag{6}$$

and its linearization at $\theta = 0$, called the **linearized pendulum equation**, is

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\theta(t) = 0. \tag{7}$$

The linearized equation is valid only for small values of $\theta(t)$, because of the assumption $\sin\theta \approx \theta$ used to obtain (7) from (6).

Damped Pendulum

Physical pendulums are subject to friction forces, which we shall assume proportional to the velocity of the pendulum. The corresponding model which includes frictional forces is called the **damped pendulum equation**:

$$\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0. \tag{8}$$

It can be written as a first order system by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)) - cy(t). \end{aligned} \tag{9}$$

Undamped Pendulum

The position-velocity differential equations for the undamped pendulum are obtained by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)). \end{aligned} \tag{10}$$

Equilibrium points of nonlinear system (10) are at $y = 0$, $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ with corresponding linearized system (see the exercises)

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\cos(n\pi)x(t). \end{aligned} \tag{11}$$

The characteristic equation of linear system (11) is $r^2 - g/L(-1)^n = 0$, because $\cos(n\pi) = (-1)^n$. The roots have different character depending on whether or not n is odd or even.

Even $n = 2m$. Then $r^2 + g/L = 0$ and the linearized system (11) is a **center**. The orbits of (11) are concentric circles surrounding $x = n\pi$, $y = 0$.

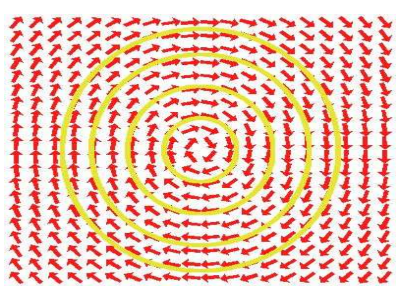

Figure 25. Linearized pendulum at $x = 2m\pi$, $y = 0$.
Orbits are concentric circles.

Odd $n = 2m + 1$. Then $r^2 - g/L = 0$ and the linearized system (11) is a **saddle**. The orbits of (11) are hyperbolas with center $x = n\pi$, $y = 0$.

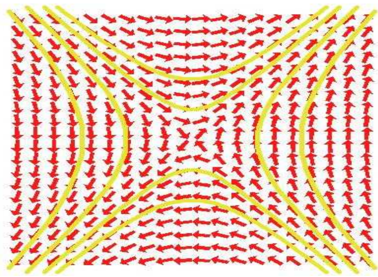

Figure 26. Linearized pendulum at $x = (2m+1)\pi$, $y = 0$.
Orbits are hyperbolas.

Drawing the Nonlinear Phase Diagram. The idea of the plot is to copy the linearized diagram onto the local region centered at the equilibrium point, when possible. The copying is guaranteed to be correct for the saddle case, but a center must be copied either as a spiral or a center. We must do extra analysis to determine the figure to copy in the case of the center.

The orbits trace an xy-curve given by integrating the separable equation

$$\frac{dy}{dx} = \frac{-g}{L}\frac{\sin x}{y}.$$

Then the conservation law for the mechanical system is

$$\frac{1}{2}y^2 + \frac{g}{L}(1 - \cos x) = E$$

where E is a constant of integration. This equation is arranged so that E is the sum of the kinetic energy $y^2/2$ and the potential energy $g(1 - \cos x)/L$, therefore E is the total mechanical energy. Using the double angle identity $\cos 2\phi = 1 - 2\sin^2\phi$ the conservation law can be written in the shorter form

$$y^2 + \frac{4g}{L}\sin^2(x/2) = 2E$$

When the energy E is small, $E < 2g/L$, then the pendulum never reaches the vertical position and it undergoes sustained periodic oscillation: the stable equilibria $(0, 2k\pi)$ have a local center structure.

When the energy E is large, $E > 2g/L$, then the pendulum reaches the vertical position and goes over the top repeatedly, represented by a saddle structure. The statement is verified from the two explicit solutions $y = \pm\sqrt{2E - 4g\sin^2(x/2)/L}$.

The energy equation $E = 2g/L$ produces the separatrices, which consist of equilibrium points plus solution curves which limit to the equilibria as $t \to \pm\infty$.

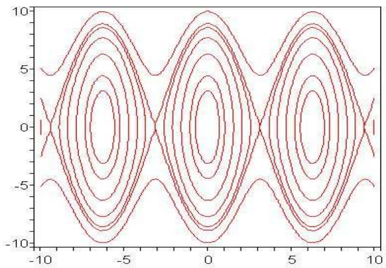

Figure 27. Nonlinear pendulum phase diagram.
Centers at $(-2\pi, 0)$, $(0, 0)$, $(2\pi, 0)$. Saddles at $(-3\pi, 0)$, $(-pi, 0)$, $(\pi, 0)$, $(3\pi, 0)$. Separatrices are generated from equilibria and $G(x, y) = 2E$, with $E = 2g/L$ and $g/L = 10$.