# Undetermined Coefficients
# The Trial Solution Method

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## Definition of Euler Solution Atom

An **Euler solution atom of a linear constant-coefficient homogeneous differential equation** is briefly called an **atom**. The set of atoms is generated from base atoms and powers of $x$.

An **Euler base atom** is one of the terms $1$, $\cos bx$, $\sin bx$, $e^{ax}$, $e^{ax}\cos bx$, $e^{ax}\sin bx$. An **Euler atom** equals $x^n$ times an Euler base atom, for $n = 0, 1, 2, 3 \ldots$.

**Examples**.
The following are atoms: $e^{2x}$, $e^{e^2 x}$, $xe^{-\pi x}$, $e^{0x}$ or $1$, $x$, $x^2$, $\cos x$, $\cos \pi x$, $e^{-x}\sin 2x$, $x^6 \sin 100x$, $x^2 e^{-5x}$, $x^5 e^{-5x}\cos 5x$, $2^x$ [equals $e^{ax}$ with $a = \ln 2$], any power $x^n$ with integer $n \geq 0$ .

The following are not atoms: $2$, $x^{-1}$, $\ln|x|$, $e^{x^2}$, $\tan x$, $\sinh x$, $\sec x$, $\csc x$, $\sin^2 x$, $\sin(x^2)$, $e^x\cos(2x+2)$, $\cot x$, $\frac{x}{1+x}$.

## Undetermined Coefficients

**Step 1**. Find a trial solution $y$ by Rule I.

> **Rule I**. Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x)$, $f'(x)$, $f''(x)$, .... Multiply these $k$ atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** $y$.
>
> > **Warning**: Rule I can **Fail**. It fails exactly when one of the atoms is a solution of the homogeneous differential equation. Apply Rule II *infra*, in case of failure of Rule I, to define trial solution $y$.

**Step 2**. Substitute trial solution $y$ into the differential equation. The resulting equation is a competition between two linear combinations of the $k$ atoms in the list.

**Step 3**. Linear independence of atoms implies matching of coefficients of atoms left and right. Write out linear algebraic equations for unknowns $d_1$, $d_2$, ..., $d_k$. Solve the equations.

**Step 4**. The trial solution $y$ with evaluated coefficients $d_1$, $d_2$, ..., $d_k$ becomes the particular solution $y_p$.

### Rule I Failure

**Example**. The differential equation $y'' = x + e^x$ has by Rule I a trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$ obtained from the list of $k = 3$ atoms $1$, $x$, $e^x$. The trial solution fails to work, because upon substitution of $y$ into the differential equation the resulting equation is

$$d_1(1)'' + d_2(x)'' + d_3(e^x)'' = 0(1) + 1(x) + 1(e^x).$$

This equation cannot be satisfied by choosing values of $d_1$, $d_2$, $d_3$, because it reads

$$x + (1 - d_3)e^x = 0,$$

implying that $x$, $e^x$ are *dependent*, a violation of the *Independence of Atoms Theorem*.

The actual trouble is a deeper problem. The equations $(1)'' = 0$ and $(x)'' = 0$ imply that $1$ and $x$ are solutions of the homogeneous differential equation $y'' = 0$. These equations cause constants $d_1$, $d_2$ to be **completely absent** from the system of equations. The constants $d_1$, $d_2$, $d_3$ must be uniquely determined. A variable that is absent in a linear system is a free variable, causing non-uniqueness, and this is the root of the problem.

## Symbols

The symbols $c_1$, $c_2$ are reserved for use as arbitrary constants in the general solution $y_h$ of the homogeneous equation. For example, the homogeneous equation $y'' + y = 0$ has general solution $y = c_1 \cos x + c_2 \sin x$.

Symbols $d_1, d_2, d_3, \ldots$ are reserved for use in the trial solution $y$ of the non-homogeneous equation. For example, the equation $y'' + y = x + e^x$ has by Rule I trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$.

## Abbreviations

$c =$ constant $=$ arbitrary constant,

$d =$ determined constant.

## Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

**(a)** Find $y_h$ as a linear combination of atoms computed by applying Euler's theorem to factors of the characteristic polynomial $ar^2 + br + c$.

**(b)** Apply the **the method of undetermined coefficients** to find $y_p$.

**Remarks**

We expect to find two arbitrary constants $c_1$, $c_2$ in the solution $y_h$, but in contrast, **no arbitrary constants** appear in $y_p$.

Calling $d_1$, $d_2$, $d_3$, $\ldots$ *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

## The Trial Solution with Fewest Euler Atoms

Undetermined coefficient theory computes a **shortest possible trial solution**, a solution with **fewest Euler atoms**.

Using the fewest atoms minimizes the size of the linear algebra problem for the constants $d_1, \ldots, d_k$. A deeper property of using the fewest atoms possible is that constants $d_1, \ldots,$ $d_k$ are *uniquely determined.*

**Example**. $y'' + y = x^2$

The atom list for $f(x) = x^2$ is $1$, $x$, $x^2$. Rule I computes a shortest trial solution $y = d_1 + d_2 x + d_3 x^2$. The linear algebra problem is $3 \times 3$, and no smaller system of equations can be found.

## The Rules for Undetermined Coefficients

**Rule I**. Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x)$, $f'(x)$, $f''(x)$, .... Multiply these $k$ atoms by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** $y$.
This rule **FAILS** if one or more of the $k$ atoms is a solution of the homogeneous differential equation.

**Rule II**. If Rule I **FAILS**, then break the $k$ atoms into groups with the same **base atom**. Cycle through the groups, replacing atoms as follows. If the first atom in the group is a solution of the homogeneous differential equation, then multiply all atoms in the group by factor $x$. Repeat until the first atom is not a solution of the homogeneous differential equation. Multiply the constructed $k$ atoms by symbols $d_1, \ldots, d_k$ and add to define trial solution $y$.

## Number of Euler Atoms in a Trial Solution

**Theorem 1 (Number of Euler Atoms)**
The number $k$ of Euler atoms computed by **Rule I** is unchanged when applying **Rule II**. Atoms changed by **Rule II** differ only by a power of $x$.

## An Illustration

Assume the constant-coefficient differential equation has order $\mathbf{2}$ and the trial solution from Rule I uses the seven (7) atoms

$$e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}, \cos x, \sin x, e^x.$$

Break the $\mathbf{7}$ atoms into $\mathbf{4}$ groups, each group with the same base atom.

| Group | Atoms | Base Atom |
|---|---|---|
| 1 | $e^{2x},\ xe^{2x},\ x^2e^{2x},\ x^3e^{2x}$ | $e^{2x}$ |
| 2 | $\cos x$ | $\cos x$ |
| 3 | $\sin x$ | $\sin x$ |
| 4 | $e^x$ | $e^x$ |

**Example 1**

Assume second order homogeneous differential equation has characteristic equation

$$(r-1)(r-3)=0.$$

Then $e^{2x}$, $\cos x$, $\sin x$ are **not** solutions of the homogeneous equation, but $e^x$ **is** a solution. The solution atom $e^{3x}$ of the homogeneous equation is not used in the trial solution construction.

**Rule I fails** because the Group 4 atom $e^x$ is a solution of the homogeneous equation. The other groups do not contain solutions of the homogeneous differential equation.
**Rule II applies** to give one new group and three unchanged groups. The trial solution $y$ is a linear combination of the $7$ atoms.

| Group | Atoms |
|---|---|
| 1 | $e^{2x}$, $xe^{2x}$, $x^2e^{2x}$, $x^3e^{ex}$ |
| 2 | $\cos x$ |
| 3 | $\sin x$ |
| New 4 | $xe^x$ |

**Details.** Atom $xe^x$ is a solution of the homogeneous equation if and only if $1$ is a double root of the characteristic equation; it isn't, which stops the multiplication by $x$ in Group 4.

**Example 2**

Assume second order homogeneous differential equation has characteristic equation

$$(r-1)(r-2)=0.$$

Then $e^x$, $e^{2x}$ are solutions of the homogeneous equation, but $\cos x$, $\sin x$ are not solutions.

**Rule I fails** because the Group 1 atom $e^{2x}$ is a solution of the homogeneous equation. **Rule II applies** to give two new groups and two unchanged groups. The trial solution $y$ is a linear combination of the $7$ atoms.

| Group | Atoms |
|---|---|
| New 1 | $xe^{2x}$, $x^2e^{2x}$, $x^3e^{2x}$, $x^4e^{ex}$ |
| 2 | $\cos x$ |
| 3 | $\sin x$ |
| New 4 | $xe^x$ |

**Details**. Atom $xe^{2x}$ is a solution of the homogeneous equation if and only if $2$ is a double root of the characteristic equation; it isn't, which stops the multiplication by $x$ in Group 1.
Atom $xe^x$ is a solution of the homogeneous equation if and only if $1$ is a double root of the characteristic equation; it isn't, which stops the multiplication by $x$ in Group 4.