1.5 Phase Line and Bifurcation Diagrams

Technical publications may use special diagrams to display **qualitative information** about the equilibrium points of the differential equation

$$y'(x) = f(y(x)). \tag{1}$$

The right side of this equation is independent of x, hence there are no external control terms that depend on x. Due to the lack of external controls, the equation is said to be **self-governing** or **autonomous**.

A **phase line diagram** for the autonomous equation $y' = f(y)$ is a line segment with labels **sink**, **source** or **node**, one for each root of $f(y) = 0$, i.e., each equilibrium; see Figure 15. It summarizes the contents of a direction field and threaded curves, including all equilibrium solutions.

The function f must be one-signed on the interval between adjacent equilibrium points, because $f(y) = 0$ means y is an equilibrium point. For this reason, a sign $+$ or $-$ is written on a phase line diagram between each pair of adjacent equilibria.

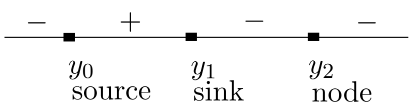

Figure 15. A phase line diagram for an autonomous equation $y' = f(y)$.

The labels are borrowed from the theory of fluids, and they have the following special definitions:[5]

Sink $y = y_0$	The equilibrium $y = y_0$ *attracts* nearby solutions at $x = \infty$: for some $H > 0$, $\lvert y(0) - y_0\rvert < H$ implies $\lvert y(x) - y_0\rvert$ decreases to 0 as $x \to \infty$.
Source $y = y_1$	The equilibrium $y = y_1$ *repels* nearby solutions at $x = \infty$: for some $H > 0$, $\lvert y(0) - y_1\rvert < H$ implies that $\lvert y(x) - y_1\rvert$ increases as $x \to \infty$.
Node $y = y_2$	The equilibrium $y = y_2$ is neither a sink nor a source.

In fluids, *sink* means fluid is lost and *source* means fluid is created. A memory device for these concepts is the kitchen sink, wherein the faucet is the *source* and the drain is the *sink*. The **stability test** below in Theorem 3 is motivated by the vector calculus results $\mathbf{Div}(\mathbf{P}) < 0$ for a sink and $\mathbf{Div}(\mathbf{P}) > 0$ for a source, where $\mathbf{P}$ is the velocity field of the fluid and **Div** is divergence.

[5] In applied literature, the special monotonic behavior required in this text's definition of a sink is relaxed to $\lim_{x\to\infty} \lvert y(x) - y_0\rvert = 0$. See page 51 for definitions of **attracting** and **repelling** equilibria.

Drawing Phase Diagrams

A phase line diagram is used to draw a **phase diagram** of threaded solutions and equilibrium solutions by using the three rules below.

1. Equilibrium solutions are horizontal lines in the phase diagram.
2. Threaded solutions of $y' = f(x, y)$ don't cross.[6] In particular, they don't cross equilibrium solutions.
3. A threaded non-equilibrium solution that starts at $x = 0$ at a point y_0 must be increasing if $f(y_0) > 0$, and decreasing if $f(y_0) < 0$.

To justify **3**, let $y_1(x)$ be a solution with $y_1'(x) = f(y_1(x))$ either positive or negative at $x = 0$. If $y_1'(x_1) = 0$ for some $x_1 > 0$, then let $c = y_1(x_1)$ and define equilibrium solution $y_2(x) = c$. Then solution y_1 crosses an equilibrium solution at $x = x_1$, violating rule **2**.

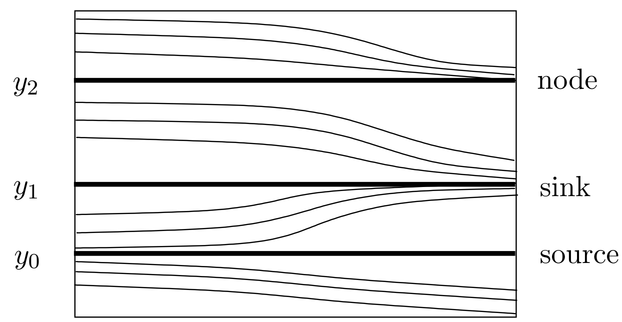

Figure 16. A phase diagram for an autonomous equation $y' = f(y)$. The graphic is drawn directly from phase line diagram Figure 15, using rules **1**, **2**, **3**. While not a replica of an accurately constructed computer graphic, the general look of threaded solutions is sufficient for intuition.

Stability Test

The terms **stable equilibrium** and **unstable equilibrium** refer to the predictable plots of nearby solutions. The term **stable** means that solutions that start near the equilibrium will stay nearby as $x \to \infty$. The term **unstable** means *not stable*. Therefore, a sink is stable and a source is unstable.

Precisely, an equilibrium y_0 is **stable** provided for given $\epsilon > 0$ there exists some $H > 0$ such that $|y(0) - y_0| < H$ implies $y(x)$ exists for $x \geq 0$ and $|y(x) - y_0| < \epsilon$.

[6] In normal applications, solutions to $y' = f(y)$ will not cross one another. Technically, this requires uniqueness of solutions to initial value problems, satisfied for example if f and f' are continuous, because of the Picard-Lindelöf theorem.

The solution $y = y(0)e^{kx}$ of the equation $y' = ky$ exists for $x \ge 0$. Properties of exponentials justify that the equilibrium $y = 0$ is a sink for $k < 0$, a source for $k > 0$ and just stable for $k = 0$.

Theorem 3 (Stability and Instability Conditions)
Let f and f' be continuous. The equation $y' = f(y)$ has a *sink* at $y = y_0$ provided $f(y_0) = 0$ and $f'(y_0) < 0$. An equilibrium $y = y_1$ is a *source* provided $f(y_1) = 0$ and $f'(y_1) > 0$. There is *no test* when f' is zero at an equilibrium. The no-test case can sometimes be decided by an additional test:

(a) Equation $y' = f(y)$ has a *sink* at $y = y_0$ provided $f(y)$ changes sign from positive to negative at $y = y_0$.

(b) Equation $y' = f(y)$ has a *source* at $y = y_0$ provided $f(y)$ changes sign from negative to positive at $y = y_0$.

Justification is postponed to page 55.

Phase Line Diagram for the Logistic Equation

The model logistic equation $y' = (1 - y)y$ is used to produce the phase line diagram in Figure 17. The logistic equation is discussed on page 6, in connection with the Malthusian population equation $y' = ky$. The letters S and U are used for stable and unstable, while N is used for a node. For computational details, see Example 30, page 54.

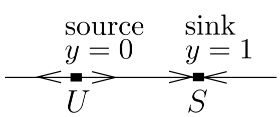

Figure 17. A phase line diagram.
The equation is $y' = (1 - y)y$. The equilibrium $y = 0$ is unstable and $y = 1$ is stable.

Arrowheads are used to display the **repelling** or **attracting** nature of the equilibrium. An equilibrium $y = y_0$ is **attracting** provided $\lim_{x\to\infty} y(x) = y_0$ for all initial data $y(0)$ with $0 < |y(0) - y_0| < h$ and $h > 0$ sufficiently small. An equilibrium $y = y_0$ is **repelling** provided $\lim_{x\to-\infty} y(x) = y_0$ for all initial data $y(0)$ with $0 < |y(0) - y_0| < h$ and $h > 0$ sufficiently small.

Direction Field Plots

A direction field for $y' = f(y)$ can be constructed in two steps. First, draw it along the y-axis. Secondly, duplicate the y-axis field at even divisions along the x-axis. Duplication is justified because $y' = f(y)$ does not depend on x, which means that the slope assigned to a lineal element at $(0, y_0)$ and (x_0, y_0) are identical.

The following facts are assembled for reference:

Fact 1. An equilibrium is a horizontal line. It is *stable* if all solutions starting near the line remain nearby as $x \to \infty$.

Fact 2. Solutions don't cross. In particular, any solution that starts above or below an equilibrium solution must remain above or below.

Fact 3. A solution curve of $y' = f(y)$ rigidly moved to the left or right will remain a solution, i.e., the translate $y(x - x_0)$ of a solution to $y' = f(y)$ is also a solution.

A phase line diagram is merely a summary of the solution behavior in a direction field. Conversely, an independently made phase line diagram can be used to enrich the detail in a direction field.

Fact 3 is used to make additional threaded solutions from an initial threaded solution, by translation. Threaded solutions with turning points are observed to have their turning points march monotonically to the left, or to the right.

Bifurcations

The phase line diagram has a close relative called a **bifurcation diagram**. The purpose of the diagram is to display qualitative information about equilibria, across all equations $y' = f(y)$, obtained by varying physical parameters appearing implicitly in f. In the simplest cases, each parameter change to $f(y)$ produces one phase line diagram and the two-dimensional stack of these phase line diagrams is the bifurcation diagram (see Figure 18).

Fish Harvesting. To understand the reason for such diagrams, consider a private lake with fish population y. The population is harvested at rate k per year. A suitable sample logistic model is

$$y' = y(4 - y) - k$$

where the constant harvesting rate k is allowed to change. Given some relevant values of k, a biologist would produce corresponding phase line diagrams, then display them by stacking, to obtain a two-dimensional diagram, like Figure 18.

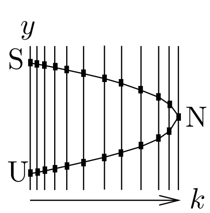

Figure 18. A bifurcation diagram.
The fish harvesting diagram consists of stacked phase-line diagrams.
Legend: U=Unstable, S=Stable, N=node.

In the figure, the vertical axis represents initial values $y(0)$ and the horizontal axis represents the harvesting rate k (axes can be swapped).

The bifurcation diagram shows how the number of equilibria and their classifications *sink*, *source* and *node* change with the harvesting rate.

Shortcut methods exist for drawing bifurcation diagrams and these methods have led to succinct diagrams that remove the phase line diagram detail. The basic idea is to eliminate the vertical lines in the plot, and replace the equilibria **dots** by a curve, essentially obtained by **connect-the-dots**. In current literature, Figure 18 is generally replaced by the more succinct Figure 19.

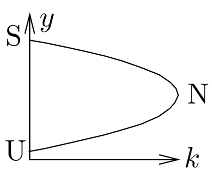

Figure 19. A succinct bifurcation diagram for fish harvesting.
Legend: U=Unstable, S=Stable, N=node.

Stability and Bifurcation Points

Biologists call a fish population *stable* when the fish reproduce at a rate that keeps up with harvesting. Bifurcation diagrams show how to stock the lake and harvest it in order to have a stable fish population.

A point in a bifurcation diagram where stability changes from stable to unstable is called a **bifurcation point**, e.g., label N in Figure 19.

The upper curve in Figure 19 gives the equilibrium population sizes of a stable fish population. Some combinations are obvious, e.g., a harvest of 2 thousand per year from an equilibrium population of about 4 thousand fish. Less obvious is a **sustainable harvest** of about 4 thousand fish with an equilibrium population of about 2 thousand fish, detected from the portion of the curve near the bifurcation point.

Harvesting rates greater than the rate at the bifurcation point will result in **extinction**. Harvesting rates less than this will also result in extinction, if the stocking size is less than the critical value realized on the lower curve in the figure. These facts are justified solely from the phase line diagram, because extinction means all solutions limit to $y = 0$.

Briefly, the lower curve gives the **minimum stocking size** and the upper curve gives the **limiting population** or **carrying capacity**, for a given harvesting rate k on the abscissa.

Examples

29 Example (No Test in Sink–Source Theorem 3) Find an example $y' = f(y)$ which has an unstable node at $y = 0$ and no other equilibria.

Solution: Let $f(y) = y^2$. The equation $y' = f(y)$ has an equilibrium at $y = 0$. In Theorem 3, there is a *no test* condition $f'(0) = 0$.

Suppose first that the nonzero solutions are known to be $y = 1/(1/y(0) - x)$, for example, by consulting a computer algebra system like `maple`:

```
dsolve(diff(y(x),x)=y(x)^2,y(x));
```

Solutions with $y(0) < 0$ limit to the equilibrium solution $y = 0$, but positive solutions "blow up" before $x = \infty$ at $x = 1/y(0)$. The equilibrium $y = 0$ is an unstable node, that is, it is not a source nor a sink.

The same conclusions are obtained from basic calculus, without solving the differential equation. The reasoning: y' has the sign of y^2, so $y' \geq 0$ and $y(x)$ increases. The equilibrium $y = 0$ behaves like a source when $y(0) > 0$. For $y(0) < 0$, again $y(x)$ increases, but in this case the equilibrium $y = 0$ behaves like a sink. Accordingly, $y = 0$ is not a source nor a sink, but a node.

30 Example (Phase Line Diagram) Verify the phase line diagram in Figure 17 for the logistic equation $y' = (1 - y)y$.

Solution: Let $f(y) = (1 - y)y$. To justify Figure 17, it suffices to find the equilibria $y = 0$ and $y = 1$, then apply Theorem 3 to show $y = 0$ is a source and $y = 1$ is a sink. The plan is to compute the equilibrium points, then find $f'(y)$ and evaluate f' at the equilibria.

$(1 - y)y = 0$	Solving $f(y) = 0$ for equilibria.
$y = 0,\ y = 1$	Roots found.
$f'(y) = (y - y^2)'$	Find f' from $f(y) = (1 - y)y$.
$\quad = 1 - 2y$	Derivative found.
$f'(0) = 1$	Positive means it is a *source*, by Theorem 3.
$f'(1) = -1$	Negative means it is a *sink*, by Theorem 3.

31 Example (Bifurcation Diagram) Verify the fish harvesting bifurcation diagram in Figure 18.

Solution: Let $f(y) = y(4 - y) - k$, where k is a parameter that controls the harvesting rate per annum. A phase line diagram is made for each relevant value of k, by applying Theorem 3 to the equilibrium points. First, the *equilibria* are computed, that is, the roots of $f(y) = 0$:

$y^2 - 4y + k = 0$	Standard quadratic form of $f(y) = 0$.
$y = \dfrac{4 \pm \sqrt{4^2 - 4k}}{2}$	Apply the quadratic formula.
$\quad = 2 + \sqrt{4 - k},\ \ 2 - \sqrt{4 - k}$	Evaluate. Real roots exist only for $4-k \geq 0$.

In preparation to apply Theorem 3, the derivative f' is calculated and then evaluated at the equilibria:

$f'(y) = (4y - y^2 - k)'$	Computing f' from $f(y) = (4 - y)y - k$.

$= 4 - 2y$	Derivative found.
$f'(2+\sqrt{4-k}) = -2\sqrt{4-k}$	Negative means a *sink*, by Theorem 3.
$f'(2-\sqrt{4-k}) = 2\sqrt{4-k}$	Positive means a *source*, by Theorem 3.

A typical phase line diagram then looks like Figure 15, page 49. In the ky-plane, sources go through the curve $y = 2 - \sqrt{4-k}$ and sinks go through the curve $y = 2 + \sqrt{4-k}$. This justifies the bifurcation diagram in Figure 18, and also Figure 19, except for the common point of the two curves at $k = 4$, $y = 2$.

At this common point, the differential equation is $y' = -(y-2)^2$. This equation is studied in Example 29, page 53; a change of variable $Y = 2 - y$ shows that the equilibrium is a node.

Proofs and Details

Stability Test Proof: Let f and f' be continuous. It will be justified that the equation $y' = f(y)$ has a *stable* equilibrium at $y = y_0$, provided $f(y_0) = 0$ and $f'(y_0) < 0$. The *unstable* case is left for the exercises.

We show that f changes sign at $y = y_0$ from positive to negative, as follows, hence the hypotheses of **(a)** hold. Continuity of f' and the inequality $f'(y_0) < 0$ imply $f'(y) < 0$ on some small interval $|y - y_0| \le H$. Therefore, $f(y) > 0 = f(y_0)$ for $y < y_0$ and $f(y) < 0 = f(y_0)$ for $y > y_0$. This justifies that the hypotheses of **(a)** apply. We complete the proof using only these hypotheses.

Global existence. It has to be established that some constant $H > 0$ exists, such that $|y(0) - y_0| < H$ implies $y(x)$ exists for $x \ge 0$ and $\lim_{x\to\infty} y(x) = y_0$. To define $H > 0$, assume $f(y_0) = 0$ and the change of sign condition $f(y) > 0$ for $y_0 - H \le y < y_0$, $f(y) < 0$ for $y_0 < y \le y_0 + H$.

Assume that $y(x)$ exists as a solution to $y' = f(y)$ on $0 \le x \le h$. It will be established that $|y(0) - y_0| < H$ implies $y(x)$ is monotonic and satisfies $|y(x) - y_0| \le Hh$ for $0 \le x \le h$.

The constant solution y_0 cannot cross any other solution, therefore a solution with $y(0) > y_0$ satisfies $y(x) > y_0$ for all x. Similarly, $y(0) < y_0$ implies $y(x) < y_0$ for all x.

The equation $y' = f(y)$ dictates the sign of y', as long as $0 < |y(x) - y_0| \le H$. Then $y(x)$ is either decreasing ($y' < 0$) or increasing ($y' > 0$) towards y_0 on $0 \le x \le h$, hence $|y(x) - y_0| \le H$ holds as long as the monotonicity holds. Because the signs endure on $0 < x \le h$, then $|y(x) - y_0| \le H$ holds on $0 \le x \le h$.

Extension to $0 \le x < \infty$. Differential equations extension theory applied to $y' = f(y)$ says that a solution satisfying on its domain $|y(x)| \le |y_0| + H$ may be extended to $x \ge 0$. This dispenses with the technical difficulty of showing that the domain of $y(x)$ is $x \ge 0$. Unfortunately, details of proof for extension results require more mathematical background than is assumed for this text; see [**?**], which justifies the extension from the Picard theorem.

It remains to show that $\lim_{x\to\infty} y(x) = y_1$ and $y_1 = y_0$. The limit equality follows because y is monotonic. The proof concludes when $y_1 = y_0$ is established.

Already, $y = y_0$ is the only root of $f(y) = 0$ in $|y - y_0| \le H$. This follows from the change of sign condition in **(a)**. It suffices to show that $f(y_1) = 0$, because then $y_1 = y_0$ by uniqueness.

To verify $f(y_1) = 0$, apply the fundamental theorem of calculus with $y'(x)$ replaced by $f(y(x))$ to obtain the identity

$$y(n+1) - y(n) = \int_n^{n+1} f(y(x))dx.$$

The integral on the right limits as $n \to \infty$ to the constant $f(y_1)$, by the integral mean value theorem of calculus, because the integrand has limit $f(y_1)$ at $x = \infty$. On the left side, the difference $y(n+1) - y(n)$ limits to $y_1 - y_1 = 0$. Therefore, $0 = f(y_1)$.

The additional test stated in the theorem is the observation that internal to the proof we used only the change of sign of f at $y = y_0$, which was deduced from the sign of the derivative $f'(y_0)$. If $f'(y_0) = 0$, but the change of sign occurs, then the details of proof still apply. The proof is complete.

Exercises 1.5

Stability-Instability Test. Find all equilibria for the given differential equation and then apply Theorem 3, page 51, to obtain a classification of each equilibrium as a **source**, **sink** or **node**. Do not draw a phase line diagram.

1. $P' = (2 - P)P$

2. $P' = (1 - P)(P - 1)$

3. $y' = y(2 - 3y)$

4. $y' = y(1 - 5y)$

5. $A' = A(A - 1)(A - 2)$

6. $A' = (A - 1)(A - 2)^2$

7. $w' = \dfrac{w(1 - w)}{1 + w^2}$

8. $w' = \dfrac{w(2 - w)}{1 + w^4}$

9. $v' = \dfrac{v(1 + v)}{4 + v^2}$

10. $v' = \dfrac{(1 - v)(1 + v)}{2 + v^2}$

Phase Line Diagram. Draw a phase line diagram, with detail similar to Figure 17.

11. $y' = y(2 - y)$

12. $y' = (y + 1)(1 - y)$

13. $y' = (y - 1)(y - 2)$

14. $y' = (y - 2)(y + 3)$

15. $y' = y(y - 2)(y - 1)$

16. $y' = y(2 - y)(y - 1)$

17. $y' = \dfrac{(y - 2)(y - 1)}{1 + y^2}$

18. $y' = \dfrac{(2 - y)(y - 1)}{1 + y^2}$

19. $y' = \dfrac{(y - 2)^2(y - 1)}{1 + y^2}$

20. $y' = \dfrac{(y - 2)(y - 1)^2}{1 + y^2}$

Bifurcation Diagram. Draw a stack of phase line diagrams and construct from it a succinct bifurcation diagram with abscissa k and ordinate $y(0)$. Don't justify details at a bifurcation point.

21. $y' = (2 - y)y - k$

22. $y' = (3 - y)y - k$

23. $y' = (2 - y)(y - 1) - k$

24. $y' = (3 - y)(y - 2) - k$

25. $y' = y(2-y)(y-1) - k$

26. $y' = y(2-y)(y-2) - k$

27. $y' = y(y-1)^2 - k$

28. $y' = y^2(y-1) - k$

29. $y' = y(0.5 - 0.001y) - k$

30. $y' = y(0.4 - 0.045y) - k$

Details and Proofs. Supply details for the following statements.

31. **(Stability Test)** Verify **(b)** of Theorem 3, page 51, by altering the proof given in the text for **(a)**.

32. **(Stability Test)** Verify **(b)** of Theorem 3, page 51, by means of the change of variable $x \to -x$.

33. **(Autonomous Equations)** Let $y' = f(y)$ have solution $y(x)$ on $a < x < b$. Then for any c, $a < c < b$, the function $z(x) = y(x+c)$ is a solution of $z' = f(z)$.

34. **(Autonomous Equations)** The method of isoclines can be applied to an autonomous equation $y' = f(y)$ by choosing equally spaced horizontal lines $y = c_i$, $i = 1, \ldots, k$. Along each horizontal line $y = c_i$ the slope is a constant $M_i = f(c_i)$, and this determines the set of invented slopes $\{M_i\}_{i=1}^k$ for the method of isoclines.