Differential Equations and Linear Algebra

2250-1 at 7:30am on 27 Apr 2012

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronic equipment are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All appended work for a chapter is expected appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on **WEBCT**. Please report recording errors by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

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Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.
Part 1. [5%]: ☐ True or ☐ False:
If the 3×3 matrices A and B are triangular, then AB is triangular.
Part 2. [5%]: ☐ True or ☐ False:
If a 3×3 matrix A has an inverse, then for all vectors $\vec{b}$,
the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

[40%] **Ch3(b)**: Determine which values of k correspond to a unique solution for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

[30%] **Ch3(c)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

[20%] **Ch3(d)**: Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the inverse of the transpose of A.

Alternate problems.

[40%] **Ch3(alt)**: This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$, a topic in Section 3.6 of Edwards-Penney.

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B^2$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

[25%] **Ch3(alt)**: Assume A is an $n \times n$ matrix and that $A\vec{x} = \vec{b}$ has a solution for any nonzero vector $\vec{b}$. Find a basis for the set S of all vectors of the form $A\vec{x}$, where $\vec{x}$ is any vector in $\mathcal{R}^n$.

[30%] **Ch3(alt)**: There are real 2×2 matrices A such that $A^2 = -4I$, where I is the identity matrix. Give an example of one such matrix A and then verify that $A^2 + 4I = 0$.

[20%] **Ch3(c3)**: Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of

$$A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Staple this page to the top of all Ch3 work.

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

[20%] **Ch4(b)**: Give an example of a matrix A with three columns that has rank 2.

[30%] **Ch4(c)**: Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

[30%] **Ch4(d)**: The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Alternate problems.

[30%] **Ch4(alt)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

[30%] **Ch4(alt)**: Consider the four vectors

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The subspaces $S_1 = \textbf{span}\{\vec{v}_1, \vec{v}_2\}$ and $S_2 = \textbf{span}\{\vec{w}_1, \vec{w}_2\}$ each have dimension 2 and share a common vector $\vec{v}_2 = \vec{w}_1$. Explain why S_1 is not equal to S_2.

[30%] **Ch4(alt)**: Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\vec{x} = \vec{0}$, where the 4×4 matrix A is given below.

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

[20%] **(alt)**: State the subspace criterion and the kernel theorem, which are the two theorems in Chapter 4 which apply to prove that a given set S in a vector space V is a subspace of V.

[25%] **Ch4(alt)**: State (1) the Wronskian test and (2) the sampling test for the independence of two functions $f_1(x)$, $f_2(x)$.

[25%] **Ch4(alt)**: Apply an independence test to the vectors below. Report **independent** or **dependent**
.

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

[25%] **Ch4(alt)**: Apply an independence test and report for which values of x the four vectors are **dependent**.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 4x \\ 3 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -2 \\ 2 \\ 9 \\ 2x \end{pmatrix}.$$

[40%] **Ch4(alt)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{4x^2 - 12x + 4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

[40%] **Ch4(alt)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{3x^2 - 14x + 3}{(x+1)^2(x-2)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$

Place this page on top of all Ch4 work.

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 11x^2 + 15e^{-x} + 3x\cos 2x$ is a solution.

[20%] **Ch5(b)**: Determine a basis of solutions of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2+r)^2((r+1)^2+7)^2 = 0.$$

[30%] **Ch5(c)**: Find the steady-state periodic solution for the equation

$$x'' + 4x' + 29x = 200\cos(t).$$

It is known that this solution equals the undetermined coefficients solution for a particular solution $x_p(t)$. This is because the homogeneous problem has roots with negative real part, which causes $\lim x_h(t) = 0$ at $t = \infty$.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 3x^2 + 4\sin 2x + 5e^x$$

Alternate problems.

[10%] **Ch5(alt)**: Report the general solution $y(x)$ of the differential equation

$$3\frac{d^3y}{dx^3} + 10\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0.$$

[20%] **Ch5(alt)**: Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 2$, $c = 2 + a$, $k = 1 + a$ and $a > 0$ a symbol, calculate all values of symbol a such that the solution $x(t)$ is **over-damped**. Please, **do not solve** the differential equation!

[40%] **Ch5(alt)**: Assume a ninth order constant-coefficient linear differential equation has characteristic equation $r^5(r^2+4)(r^2+r) = 0$. Suppose the right side of the differential equation is

$$f(x) = x^2(x + 2e^{-x}) + 5\sin 2x + x\cos x + 7e^x$$

Determine the **shortest** trial solution for a particular solution y_p according to the method of undetermined coefficients. To save time, **do not evaluate** the undetermined coefficients!

[20%] **Ch5(alt)**: A particular solution of the differential equation $x'' + 2x' + 17x = 50\cos(3t)$ is

$$x(t) = 4\cos 3t + 12e^{-t}\sin 4t + 3\sin 3t + 15e^{-t}\cos 4t.$$

Identify the **steady-state** solution $x_{\text{ss}}(t)$ and the **transient** solution $x_{\text{tr}}(t)$.

[15%] **Ch5(alt)**: Find the general solution, given characteristic equation

$$(r^2+2r)^2(r^4-4r^2)(r^2+4r+8) = 0.$$

[25%] **Ch5(alt)**: Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

1.[12%] $r^3(r^2 - 5r)^2(r^2 - 25) = 0,$
2.[13%] $(r - 4)^2(r^2 + 2r + 3)^2(r^2 - 16)^3 = 0$

[25%] **Ch5(alt)**: Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 10$, $c = 13$ and $k = 4$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [5%].

[50%] **Ch5(alt)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv} + 9y'' = x(x + 2e^{3x}) + 3x\cos 3x + 4e^{-3x}$$

Place this page on top of all Ch5 work.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[20%] **Ch6(a)**: Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

[40%] **Ch6(b)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$.
To save time, **don't find the eigenvector for** $\lambda = 0$.

Alternate problems.

[25%] **Ch6(alt)**: Let A be a 2×2 matrix satisfying for all real numbers c_1, c_2 the identity

$$A\left(c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) = 4c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $AP = PD$.

[40%] **Ch6(alt)**: Find the two eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

[30%] **Ch6(alt)**: Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

[20%] **Ch6(alt)**: Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Ch6(alt): Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

[35%] **Ch6(alt)**: The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

[20%] **Ch6(alt)** Assume A is a given 4×4 matrix with eigenvalues 0, 1, $3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

[40%] **Ch6(alt)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

[20%] **Ch6(alt)**: Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

[25%] **Ch6(alt)**: Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

☐ [25%] **Ch6(alt)**: Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

[25%] **Ch6(alt)**: Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

[25%] **Ch6(alt)**: Let A be a 3×3 matrix with eigenpairs

$$(5, \vec{v}_1), \quad (3, \vec{v}_2), \quad (-1, \vec{v}_3).$$

Let $P = \mathbf{aug}(\vec{v}_2, \vec{v}_1, \vec{v}_3)$. Display the answer for $P^{-1}AP$ [20%]. Justify your claim with a sentence [5%].

[25%] **Ch6(alt)**: Let A be a 2×2 matrix with eigenpairs

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2).$$

Display Fourier's model for the matrix A.

Place this page on top of all Ch6 work.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[50%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned} \frac{dx}{dt} &= x + 3y, \\ \frac{dy}{dt} &= 18x + 4y. \end{aligned}$$

[50%] **Ch7(b)**: Define

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

The eigenvalues of A are 3, 3, 6. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\vec{u}' = A\vec{u}$. Show all eigenanalysis steps and display the differential equation answer $\vec{u}(t)$ in vector form.

Alternate problems.

[30%] **Ch7(alt)**: Let A be an $n \times n$ matrix of real numbers. State three different methods for solving the system $\vec{\vec{u}}' = A\vec{\vec{u}}$, which you learned in this course.

[15%] **Ch7(alt)**: Solve the 3×3 differential system $\vec{u}' = A\vec{u}$ for matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

[20%] **Ch7(c)**: Find the 2×2 matrix A, given the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

[15%] **Ch7(alt)**: A 2×2 real matrix A has eigenvalues -1 and -2. Display the form of the general solution of the differential equation $\vec{u}' = A\vec{u}$.

[25%] **Ch7(alt)**: Give an example of a 2×2 real matrix A for which Fourier's model is not valid. Then display the general solution $\vec{x}(t)$ of $\vec{x}' = A\vec{x}$.

[25%] **Ch7(alt)**: Consider a 2×2 system $\vec{x}' = A\vec{x}$. Assume A has complex eigenvalues $\lambda = \pm\sqrt{3}i$. Prove that $\lim_{t\to\infty} |\vec{x}(t)| = \infty$ is false for every possible solution $\vec{x}(t)$.

[25%] **Ch7(alt)**: Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 10$ gallons/minute. Let A empty o B at rate r, and let B empty at rate r. Assume the model brine tank mode

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 10, \quad y(0) = 15. \end{cases}$$

Find the maximum amount of salt ever in tank B for $t \geq 0$.

[25%] **Ch7(alt)**: A 3×3 real matrix A has all eigenvalues equal to zero and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Find the general solution of the differential equation $\vec{x}' = A\vec{x}$.

[50%] **Ch7(alt)**: Apply the eigenanalysis method to solve the system $\vec{x}' = A\vec{x}$, given

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

[25%] **Ch7(alt)**: Solve for $x(t)$ in the system below. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + y, \\ y' &= -9x + y. \end{aligned}$$

[25%] **Ch7(alt)**: Consider a 4×4 system $\vec{x}' = A\vec{x}$. Assume A has an eigenvalue $\lambda = -1/7$ with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find a nonzero solution of the differential equation with limit zero at infinity.

[25%] **Ch7(alt)**: Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 5, \quad y(0) = 10. \end{cases}$$

Find an equation for the amount of salt in tank B.

Place this page on top of all Ch7 work.

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Ch8. (Matrix Exponential) Complete all problems.

[40%] **Ch8(a)**: Using any method in the lectures or the textbook, display the matrix exponential e^{Bt}, for

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

[30%] **Ch8(b)**: Consider the 2×2 system

$$\begin{aligned} x' &= 3x, \\ y' &= -y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system as a matrix problem $\vec{u}' = A\vec{u}$ for $\vec{u}$, using the matrix exponential e^{At}.

[30%] **Ch8(c)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= 3x + 3, \\ y' &= -y + 1. \end{aligned}$$

Alternate problems.

[30%] **Ch8(alt)**: Check the correct statements.

☐ **1**. The general solution of $\vec{u}' = A\vec{u}$ is a vector linear combination of atoms found from the roots of $\det(A - \lambda I) = 0$.

☐ **2**. The system $\vec{u}' = A\vec{u}$ can only be solved when A is diagonalizable.

☐ **3**. The general solution of $\vec{u}' = A\vec{u}$ can be written as $\vec{u}(t) = e^{At}\vec{u}(0)$.

☐ **4**. The matrix exponential e^{At} can be found using Laplace theory.

☐ **5**. For any $n \times n$ matrix A, $(sI - A)^{-1}$ equals the Laplace integral of e^{At}.

☐ **6**. A second order system $\vec{x}'' = A\vec{x} + \vec{G}(t)$ can be transformed into a first order system of the form $\vec{u}' = B\vec{u} + \vec{F}(t)$.

Place this page on top of all Ch8 work.

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 - y + 9, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 3$, $y = 3$. Compute the Jacobian matrix $A = J(3, 3)$ of the linearized system at this equilibrium point.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x + 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{aligned}$$

At equilibrium point $x = -3$, $y = 3$, the Jacobian matrix is $A = J(-3, 3) = \begin{pmatrix} 10 & 4 \\ 3 & 3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\vec{u}' = A\vec{u}$.

(2) Apply a theorem to classify $x = -3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem.

Place this page on top of all Ch9 work.

Name ______________ **7:30am**

Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{1}{s^3}$	$\dfrac{1}{s-4}$	$\dfrac{s}{s^2+4}$	$\dfrac{2}{s^2+1}$	$\dfrac{e^{-s}}{s+1}$

$f(t)$	t	te^t	$t\cos 2t$	$t(1+e^t)$	$e^t\sin t$
$\mathcal{L}(f(t))$					

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for the pulse $f(t) = t$ on $1 \le t < 2$, $f(t) = 0$ otherwise.

[30%] **Ch10(c)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - 2x'(t) = 4e^{2t}, \quad x(0) = x'(0) = 0.$$

Alternate problems.

[20%] **Ch10(alt)**: Let $f(t) = |\sin t|$ on $0 \le t \le 2\pi$, with $f(t)$ periodic of period 2π. Display the formula for $\mathcal{L}(f(t))$ according to the periodic function rule. To save time, **don't evaluate any integrals**.

[20%] **Ch10(alt)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-s}}{s-2}$.

[20%] **Ch10(alt)**: Let $f(t) = |\sin t|$ on $0 \le t \le 2\pi$, with $f(t)$ periodic of period 2π. Display the formula for $\mathcal{L}(f(t))$ according to the periodic function rule. To save time, **don't evaluate any integrals**.

[20%] **Ch10(alt)**: Solve for $f(t)$ in the equation $\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{1}{(s+1)^2} + \dfrac{d^2}{ds^2}\mathcal{L}(\sin t)$.

[40%] **Ch10(alt)** Use Laplace's method to find an explicit formula for $x(t)$. Don't find $y(t)$!

$$\begin{aligned} x'(t) &= 2x(t) + 5y(t), \\ y'(t) &= 5x(t) + 2y(t), \\ x(0) &= 1, \\ y(0) &= 1. \end{aligned}$$

[25%] **Ch10(alt)**: Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

[15%] **Ch10(alt)**: Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{3t}\right)\Big|_{s\to(s+3)}\right).$$

[20%] **Ch10(alt)**: Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

[15%] **Ch10(alt)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

[25%] **Ch10(alt)**: Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\dfrac{\sin(t)}{t}$.

[20%] **Ch10(alt)**: Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{s+2}$	$\frac{s+1}{s^2+2s+5}$		

[30%] **Ch10(alt)**: Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

[35%] **Ch10(alt)**: Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t+\sin t)|_{s\to(s-2)}.$$

[30%] **Ch10(alt)**: Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

[30%] **Ch10(alt)**: Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} - x'' = 3t^2 + 4e^{-2t} + 5e^t\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Place this page on top of all Ch10 work.