Differential Equations and Linear Algebra

2250-1 at 7:30am on 27 Apr 2012

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronic equipment are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All appended work for a chapter is expected appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on **WEBCT**. Please report recording errors by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

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Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.
Part 1. [5%]: ☐ True or ☐ False:
If the 3×3 matrices A and B are triangular, then AB is triangular.
Part 2. [5%]: ☐ True or ☐ False:
If a 3×3 matrix A has an inverse, then for all vectors $\vec{b}$,
the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

Answer: False. True.

[40%] **Ch3(b)**: Determine which values of k correspond to a unique solution for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer: There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 2$ and $k \neq 3$. Alternative solution: Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$.
Then (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 2$.

[30%] **Ch3(c)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

Answer: $x_2 = \Delta_2/\Delta$, $\Delta_2 = \det \begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & -2 \end{pmatrix} = 36$, $\Delta = \det(A) = 4$, $x_2 = 9$.

[20%] **Ch3(d)**: Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the inverse of the transpose of A.

Answer: Then $(A^T)^{-1} = (A^{-1})^T = \left(\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}\right)^T = \begin{pmatrix} 2 & 0 \\ -6 & 4 \end{pmatrix}$.

Alternate problems.

[40%] **Ch3(alt)**: This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$, a topic in Section 3.6 of Edwards-Penney.

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B^2$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Answer: The determinant of $C^{-1}B^2$ is $|B|^2/|C|$. Then $CB == \mathbf{adj}(B)B = |B|I$ implies $|C||B| = \det(|B|I) = |B|^4$. Because $|C| = |B|^3$, then the answer is $1/|B|$. Return to $CB = |B|I$ and do one dot product to find the value $|B| = 8$. We report $\det(C^{-1}B^2) = 1/|B| = 1/8$.

[25%] **Ch3(alt)**: Assume A is an $n \times n$ matrix and that $A\vec{x} = \vec{b}$ has a solution for any nonzero vector $\vec{b}$. Find a basis for the set S of all vectors of the form $A\vec{x}$, where $\vec{x}$ is any vector in $\mathcal{R}^n$.

Answer: The basis could be the columns of the identity matrix.

[30%] **Ch3(alt)**: There are real 2×2 matrices A such that $A^2 = -4I$, where I is the identity matrix. Give an example of one such matrix A and then verify that $A^2 + 4I = 0$.

Answer: Choose any matrix whose characteristic equation is $\lambda^2 + 4 = 0$. Then $A^2 + 4I = 0$ by the Cayley-Hamilton theorem.

[20%] **Ch3(c3)**: Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of

$$A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Answer: The answer is the cofactor of A in row 4, column 3 $= (-1)^7$ times minor of A in 4,3 $= -2$.

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Answer: The first and fourth apply to the given functions, while the others apply only to fixed vectors.

[20%] **Ch4(b)**: Give an example of a matrix A with three columns that has rank 2.

Answer: Let $\vec{v}_1$, $\vec{v}_2$ be columns of the identity and let $\vec{v}_3$ be the zero vector. Define A to be the augmented matrix of these three vectors. Then A has 3 columns and rank 2.

[30%] **Ch4(c)**: Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the kernel theorem. This is theorem 2 in section 4.2 of Edwards-Penney. Then S is a subspace of $\mathcal{R}^3$. Another possible solution: The kernel theorem (theorem 2 in 4.2, Edwards-Penney) applies, because the given restriction equations are linear homogeneous algebraic equations. Therefore, S is a subspace of $\mathcal{R}^3$.

[30%] **Ch4(d)**: The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Answer: Find $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 1 and 4.

Alternate problems.

[30%] **Ch4(alt)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer: Independent. The rank of the augmented matrix of the three vectors is 3.

[30%] **Ch4(alt)**: Consider the four vectors

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

The subspaces $S_1 = \mathbf{span}\{\vec{v}_1, \vec{v}_2\}$ and $S_2 = \mathbf{span}\{\vec{w}_1, \vec{w}_2\}$ each have dimension 2 and share a common vector $\vec{v}_2 = \vec{w}_1$. Explain why S_1 is not equal to S_2.

Answer: Because v_1, v_2, w_2 are independent by the determinant test. This means w_2 is not in S_1.

[30%] **Ch4(alt)**: Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\vec{x} = \vec{0}$, where the 4×4 matrix A is given below.

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Answer: $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, basis $= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$

[20%] **(alt)**: State the subspace criterion and the kernel theorem, which are the two theorems in Chapter 4 which apply to prove that a given set S in a vector space V is a subspace of V.

Answer: (1) A subset S of a vector space V is a subspace of V provided the following three items hold: (a) The zero vector is in S; (b) If x_1, x_2 are in S, then $x_1 + x_2$ is in S; (c) If x is in S and c is a constant, then cx is in S. (2) The kernel theorem assumes that $V = \mathcal{R}^n$ and S is the set of solutions for a set of linear homogeneous algebraic equations (i.e., a system $A\vec{x} = \vec{0}$ in vector-matrix notation). The conclusion is that S is a subspace. Set S is called the kernel of the system of equations or the kernel of the matrix A. The words nullspace and kernel mean the same thing.

[25%] **Ch4(alt)**: State (1) the Wronskian test and (2) the sampling test for the independence of two functions $f_1(x)$, $f_2(x)$.

Answer: (1) Let $W = \begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix}$. If $det(W) \neq 0$ for some $x = x_0$, then the two functions are independent. (2) Choose two sample values $x = x_1, x_2$. Let $W = \begin{pmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{pmatrix}$. If $\det(W) \neq 0$, then the two functions are independent.

[25%] **Ch4(alt)**: Apply an independence test to the vectors below. Report **independent** or **dependent**
.

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer: The determinant test does not apply directly, but the rank test does apply. The rank is 2, found by computing the reduced row echelon form or the augmented matrix of the three vectors. We report **dependent**.

[25%] **Ch4(alt)**: Apply an independence test and report for which values of x the four vectors are **dependent**.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 4x \\ 3 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -2 \\ 2 \\ 9 \\ 2x \end{pmatrix}.$$

Answer: Dependent when $2x = 6, 2x = 5/2$, by the determinant test. The vectors are dependent if and only if the determinant is zero.

[40%] **Ch4(alt)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{4x^2 - 12x + 4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

Answer: Clear the fractions, substitute samples for x to get four equations in four unknowns. Solve the equations. Maybe $b = -1, d = 5, a = c = 0$?

[40%] **Ch4(alt)**: Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{3x^2 - 14x + 3}{(x+1)^2(x-2)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$

Answer: Clear the fractions, substitute samples for x to get four equations in four unknowns. Solve the equations. Maybe $a = -20/27, b = 20/9, c = 20/27, d = -13/9$?

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 11x^2 + 15e^{-x} + 3x\cos 2x$ is a solution.

Answer: The atoms x^2, e^{-x}, $x\cos 2x$ correspond to roots $0, 0, 0, -1, 2i, -2i, 2i, -2i$. The root-factor theorem of college algebra implies the characteristic polynomial should be $r^3(r+1)(r^2+4)^2$.

[20%] **Ch5(b)**: Determine a basis of solutions of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2 + r)^2((r + 1)^2 + 7)^2 = 0.$$

Answer: The roots are $0, 0, 0, -1, -1, -1\pm\sqrt{7}i, -1\pm\sqrt{7}i$. By Euler's theorem, a basis is the set of atoms for these roots: $1, x, x^2, e^{-x}, xe^{-x}, e^{-x}\cos(\sqrt{7}x), e^{-x}\sin(\sqrt{7}x), xe^{-x}\cos(\sqrt{7}x), xe^{-x}\sin(\sqrt{7}x)$.

[30%] **Ch5(c)**: Find the steady-state periodic solution for the equation

$$x'' + 4x' + 29x = 200\cos(t).$$

It is known that this solution equals the undetermined coefficients solution for a particular solution $x_p(t)$. This is because the homogeneous problem has roots with negative real part, which causes $\lim x_h(t) = 0$ at $t = \infty$.

Answer: Use undetermined coefficients trial solution $x = d_1\cos t + d_2\sin t$. Then $d_1 = 7$, $d_2 = 1$ and $x_p(t) = 7\cos t + \sin t$ is the steady-state periodic solution.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 3x^2 + 4\sin 2x + 5e^x$$

Answer: Let $f(x) = 3x^2 + 4\sin 2x + 5e^x$. The atoms in f are x^2, $\sin 2x$, e^x. The complete set of distinct atoms appearing in the derivatives $f, f', f'', \ldots$ is $1, x, x^2, \cos 2x, \sin 2x, e^x$. There are 6 atoms in this list. A theorem says that the shortest trial solution contains 6 atoms. Break the 6 atoms into four groups, each with the same base atom: group 1 $==$ $1, x, x^2$; group 2 $==$ $\cos 2x$; group 3 $==$ $\sin 2x$; group 4 $==$ e^x. Modify each group, by multiplication by x until the group contains no solution of the homogeneous equation $y^{(3)} - y^{(2)} = 0$. Then the four groups are replaced by group 1^* $==$ x^2, x^3, x^4; group 2^* $==$ $\cos 2x$; group 3^* $==$ $\sin 2x$; group 4^* $==$ xe^x. The shortest trial solution is a linear combination of these last six atoms.

Alternate problems.

[10%] **Ch5(alt)**: Report the general solution $y(x)$ of the differential equation

$$3\frac{d^3y}{dx^3} + 10\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0.$$

Answer: Characteristic equation $r(3r + 1)(r + 3) = 0$ has roots 0, $-1/3$, -3. The general solution $y(x)$ is a linear combination of the atoms 1, $e^{-x/3}$, e^{-3x}.

[20%] **Ch5(alt)**: Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=2$, $c=2+a$, $k=1+a$ and $a>0$ a symbol, calculate all values of symbol a such that the solution $x(t)$ is **over-damped**. Please, **do not solve** the differential equation!

> **Answer:** The discriminant must be positive. Then $(2+a)^2-8(1+a)>0$ or $a^2-4a-4>0$.

[40%] **Ch5(alt)**: Assume a ninth order constant-coefficient linear differential equation has characteristic equation $r^5(r^2+4)(r^2+r)=0$. Suppose the right side of the differential equation is

$$f(x)=x^2(x+2e^{-x})+5\sin 2x+x\cos x+7e^x$$

Determine the **shortest** trial solution for a particular solution y_p according to the method of undetermined coefficients. To save time, **do not evaluate** the undetermined coefficients!

> **Answer:** The homogeneous problem has roots $0,0,0,0,0,0,-1,\pm 2i$ with atoms $1,x,x^2,x^3,x^4,x^5,e^{-x},\cos 2x$, si
> The trial solution is constructed initially from $f(x)=x^3+2x^2e^{-x}+5\sin 2x+x\cos x+7e^x$, which has atom list in 7 groups (1) $1,x,x^2,x^3$; (2) e^{-x},xe^{-x},x^2e^{-x}; (3) e^x; (4) $\cos 2x$; (5) $\sin 2x$; (6) $\cos x, x\cos x$; (7) $\sin x, x\sin x$. Conflicts with the homogeneous equation atoms causes a repair of groups (1), (2), (4), (5), making the new groups (1) x^6,x^7,x^8,x^9; (2) $xe^{-x},x^2e^{-x},x^3e^{-x}$; (3) e^x; (4) $x\cos 2x$; (5) $x\sin 2x$; (6) $\cos x,x\cos x$; (7) $\sin x,x\sin x$. Then the trial solution is a linear combination of the 14 atoms in the corrected list.

[20%] **Ch5(alt)**: A particular solution of the differential equation $x''+2x'+17x=50\cos(3t)$ is

$$x(t)=4\cos 3t+12e^{-t}\sin 4t+3\sin 3t+15e^{-t}\cos 4t.$$

Identify the **steady-state** solution $x_{\text{SS}}(t)$ and the **transient** solution $x_{\text{tr}}(t)$.

> **Answer:** The transient solution is the sum of all terms with limit zero. The steady-state is the sum of the remaining terms. Then $x_{\text{SS}}=4\cos 3t+3\sin 3t$ and $x_{\text{tr}}=12e^{-t}\sin 4t+15e^{-t}\cos 4t$.

[15%] **Ch5(alt)**: Find the general solution, given characteristic equation

$$(r^2+2r)^2(r^4-4r^2)(r^2+4r+8)=0.$$

> **Answer:** Roots $0,0,0,0,-2,-2,-2,2,-2\pm 2i$, atoms $1,x,x^2,x^3,e^{-2x},xe^{-2x},e^x,e^{-2x}\cos 2x,e^{-2x}\sin 2x$, and $y=$ a linear combination of the atoms.

[25%] **Ch5(alt)**: Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

$$\begin{array}{ll} \textbf{1.}[12\%] & r^3(r^2-5r)^2(r^2-25)=0, \\ \textbf{2.}[13\%] & (r-4)^2(r^2+2r+3)^2(r^2-16)^3=0 \end{array}$$

> **Answer:** **1.** Roots $=0,0,0,0,0,5,5,5,-5$, atoms $1,x,x^2,x^3,x^4,e^{5x},xe^{5x},x^2e^{5x},e^{-5x}$, $y=$ a linear combination of the atoms. **2.** Roots $=2,2,-2,-2,-1\pm\sqrt{2},4,4,4,-4,-4,-4$. There are 12 atoms $e^{2x},xe^{2x},e^{-2x},xe^{-2x},e^{-x+\sqrt{2}x},e^{-x-\sqrt{2}x},e^{4x},xe^{4x},x^2e^{4x},e^{-4x},xe^{-4x},x^2e^{-4x}$. Then $y=$ a linear combination of the atoms.

[25%] **Ch5(alt)**: Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=10$, $c=13$ and $k=4$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [5%].

> **Answer:** Factor the characteristic equation into $(2r+1)(5r+4)$ with distinct real roots $-1/2,-4/5$. Then the atoms are $e^{-t/2},e^{-4t/5}$ and $y=$ a linear combination of the atoms. Classified as over-damped.

[50%] **Ch5(alt)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv} + 9y'' = x(x + 2e^{3x}) + 3x\cos 3x + 4e^{-3x}$$

Answer: The homogeneous problem has roots $0, 0, \pm 3i$ with atoms $1, x, \cos 3x, \sin 3x$. The trial solution is constructed initially from $f(x) = x^2 + 2xe^{3x} + 3x\cos 3x + 4e^{-3x}$, which has seven atoms in a list in four groups (1) $1, x, x^2$; (2) e^{3x}, xe^{3x}; (3) $\cos 3x$; (4) $\sin 3x$. Conflicts with the homogeneous equation atoms causes a repair of groups (1), (3), (4), making the new groups (1) x^2, x^3, x^4; (2) e^{3x}, xe^{3x}; (3) $x\cos 3x$; (4) $x\sin 3x$. Then the shortest trial solution is a linear combination of the seven atoms in the corrected list.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[20%] **Ch6(a)**: Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer: The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

[40%] **Ch6(b)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer: The characteristic polynomial is $\det(A - rI) = (-r)(3-r)(r-2)^2$. The eigenvalues are $0, 2, 2, 3$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$.
To save time, **don't find the eigenvector for** $\lambda = 0$.

Answer: Because $A - 2I = \begin{pmatrix} -2 & -12 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 1 & 0 & -15/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then there is only one eigenpair for $\lambda = 2$, with eigenvector $\vec{v} = \begin{pmatrix} 15 \\ -2 \\ 2 \end{pmatrix}$.

Alternate problems.

[25%] **Ch6(alt)**: Let A be a 2×2 matrix satisfying for all real numbers c_1, c_2 the identity

$$A\left(c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) = 4c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $AP = PD$.

Answer: Choose $c_1 = 1, c_2 = 0$ in the identity to get $A\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This equation means 0 is an eigenvalue, giving first eigenpair $\left(0, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$. The second eigenpair $\left(4, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right)$

is found the same way, choosing $c_1 = 0, c_2 = 1$. Then $D = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ (diagonal entries are the eigenvalues $0, 4$) and $P = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ (columns are the eigenvectors, in the same order).

[40%] **Ch6(alt)**: Find the two eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Answer: $\left(-1 + 2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$, $\left(-1 - 2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

[30%] **Ch6(alt)**: Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer: The first and the last, because the test $A\vec{x} = \lambda\vec{x}$ passes in both cases.

[20%] **Ch6(alt)**: Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Answer: Both B and C have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both B and C are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \mathbf{diag}(-1, 1, 5)$. In this case, $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 15) = \mathbf{diag}(-1, 5, 17)$ and the eigenvalues of A are $-1, 5, 17$.

Ch6(alt): Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Answer: Because $AP = PD$, then $D = P^{-1}AP$ is the diagonal matrix of eigenvalues, taken in the order determined by the eigenpairs $(3, \vec{v}_2), (1, \vec{v}_3), (4, \vec{v}_1)$. Then $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

[35%] **Ch6(alt)**: The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Answer: Compute $\mathbf{rref}(A - 3I) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This has rank 2, nullity 1. There is just one eigenvector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. No Fourier's model, not diagonalizable.

[20%] **Ch6(alt)** Assume A is a given 4×4 matrix with eigenvalues $0,\ 1,\ 3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

Answer: Such a matrix is diagonalizable, because of four distinct eigenvalues. Then $4B - 3I$ has the same eigenvalues for all matrices B similar to A. In particular, $4A - 3I$ has the same eigenvalues as $4D - 3I$ where D is the diagonal matrix with entries $0, 1, 3 + 2i, 3 - 2i$. Compute $4D - 3I = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9+8i & 0 \\ 0 & 0 & 0 & 9-8i \end{pmatrix}$. The answer is $0, 1, 9 + 8i, 9 - 8i$.

[40%] **Ch6(alt)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer: The characteristic polynomial is $\det(A-rI) = (r^2+6)(3-r)(r-2)^2$. The eigenvalues are $2, 2, 3, \pm\sqrt{6}i$. Determinant expansion is by the cofactor method along column 5. This reduces it to a 4×4 determinant, which can be expanded as a product of two quadratics. In detail, we first get $|A - rI| = (3 - r)|B - rI|$, where $B = \begin{pmatrix} 0 & -2 & -5 & 0 \\ 3 & 0 & -12 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. So we have one eigenvalue 3, and we find the eigenvalues of B. Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer for the eigenvalues of A is 3 and the eigenvalues of B_1 and B_3. We report $3, \pm\sqrt{6}i, 2, 2$. It is also possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

[20%] **Ch6(alt)**: Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

Answer: (1) $P = \begin{pmatrix} 13 & i & -i \\ 6 & 2 & 2 \\ -41 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$. (2) $AP = PD$ implies $A = PDP^{-1}$.

[25%] **Ch6(alt)**: Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

Answer: Because the answer is the same for all matrices similar to A (that is, all $B = PAP^{-1}$) then it suffices to answer the question for diagonal matrices. We know A is diagonalizable, because it has distinct eigenvalues. So we choose D equal to the diagonal matrix with entries $2, 3, 4$. Compute $\frac{1}{3}D - 2I = \begin{pmatrix} \frac{2}{3} - 2 & 0 & 0 \\ 0 & \frac{3}{3} - 2 & 0 \\ 0 & 0 & \frac{4}{3} - 2 \end{pmatrix}$. Then the eigenvalues are $-\frac{4}{3}, -1, -\frac{2}{3}$.

☐ [25%] **Ch6(alt)**: Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

Answer: The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then A and B have exactly the same characteristic equation, hence exactly the same eigenvalues.

[25%] **Ch6(alt)**: Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Answer: The characteristic polynomial is $\det(B - rI) = (2 - r)(5 - r)(5 - r)(3 - r)$. The eigenvalues are $2, 3, 5, 5$.

It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3. These are quickly found by Sarrus' Rule applied to the two 2×2 determinants $|B_1 - rI| = (2 - r)(5 - r)$ and $|B_3 - rI| = r^2 - 8r + 15 = (5 - r)(3 - r)$.

[25%] **Ch6(alt)**: Let A be a 3×3 matrix with eigenpairs

$$(5, \vec{v}_1), \quad (3, \vec{v}_2), \quad (-1, \vec{v}_3).$$

Let $P = \mathbf{aug}(\vec{v}_2, \vec{v}_1, \vec{v}_3)$. Display the answer for $P^{-1}AP$ [20%]. Justify your claim with a sentence [5%].

Answer: From $AP = PD$ we have $D = P^{-1}AP$. The diagonal entries of D are $3, 5, -1$.

[25%] **Ch6(alt)**: Let A be a 2×2 matrix with eigenpairs

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2).$$

Display Fourier's model for the matrix A.

Answer: $A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$ where c_1, c_2 are arbitrary constants.

Place this page on top of all Ch6 work.

Name ______________ **7:30am**

Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Ch7. (Linear Systems of Differential Equations) Complete all problems.

[50%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned} \frac{dx}{dt} &= x + 3y, \\ \frac{dy}{dt} &= 18x + 4y. \end{aligned}$$

Answer: Define $A = \begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix}$. The eigenvalues $-5, 10$ are roots of the characteristic equation $\det(A - rI) = (r+5)(r-10) = 0$. By Cayley-Hamilton-Zeibur, $x(t) = c_1 e^{-5t} + c_2 e^{10t}$. Using the first differential equation $x' = x + 3y$ implies $3y(t) = x' - x = -6c_1 e^{-5t} + 9c_2 e^{10t}$. Then $y(t) = -2c_1 e^{-5t} + 3c_2 e^{10t}$.

Eigenanalysis would use the eigenpairs $\left(-5, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$, $\left(10, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$ to report solution $\vec{u}(t) = c_1 \vec{v}_1 e^{-5t} + c_2 \vec{v}_2 e^{10t}$.

Laplace theory would start with the resolvent equation $(sI - A)\mathcal{L}(\vec{u}(t)) = \vec{u}(0)$. Then solve for $\mathcal{L}(x(t)), \mathcal{L}(y(t))$ and finally use the backward table and Lerch's theorem to find the general solution in terms of constants c_1, c_2 defined by $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

[50%] **Ch7(b)**: Define

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

The eigenvalues of A are 3, 3, 6. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\vec{u}' = A\vec{u}$. Show all eigenanalysis steps and display the differential equation answer $\vec{u}(t)$ in vector form.

Answer: Form the matrix $A_1 = A - (3)I = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$. Then $\mathbf{rref}(A_1) = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies the general solution of $A_1\vec{x} = \vec{0}$ is $x_1 = t_1/2 + t_2/2$, $x_2 = t_1$, $x_3 = t_2$. Take the partials on symbols t_1, t_2 to obtain the eigenvectors $\vec{v}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$. Repeat with $A_2 = A - (6)I = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -3 & 0 \\ -2 & 1 & -2 \end{pmatrix}$. Then $\mathbf{rref}(A_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies the general solution of $A_2\vec{x} = \vec{0}$ is $x_1 = -t_1$, $x_2 = 0$, $x_3 = t_1$. Take the partial on symbol t_1 to obtain the eigenvector $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The general solution of $\vec{u}' = A\vec{u}$ is $\vec{u}(t) = c_1 e^{3t}\vec{v}_1 + c_2 e^{3t}\vec{v}_2 + e^{6t}\vec{v}_3$.

Alternate problems.

[30%] **Ch7(alt)**: Let A be an $n \times n$ matrix of real numbers. State three different methods for solving the system $\vec{u}' = A\vec{u}$, which you learned in this course.

Answer: Eigenanalysis, Cayley-Hamilton-Ziebur, Laplace resolvent, exponential matrix.

[15%] **Ch7(alt)**: Solve the 3×3 differential system $\vec{u}' = A\vec{u}$ for matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Answer: The matrix is triangular, so the theory of linear cascades applies. First, $x_3' = 0$, then $x_3 = c_1$. Back-substitute: $x_2' = x_3 = c_1$. Then $x_2 = c_1 t + c_2$. Finally, $x_1' = x_2 = c_1 t + c_2$, which implies $x_1 = c_1 t^2/2 + c_2 t + c_3$.

Does Cayley-Hamilton-Ziebur work? Yes, it always works. The roots of $|A - rI| = 0$ are $r = 0, 0, 0$, which implies atoms $1, t, t^2$ and then $\vec{u} = \vec{d}_1 + \vec{d}_2 t + \vec{d}_3 t^2$. Differentiate this relation two times and set $t = 0$ in the resulting 3 equations in 3 vector unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$. Solve the system for the unknown vectors, in terms of A and $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \vec{c}$. You will get $\vec{c} = \vec{d}_1, A\vec{c} = \vec{d}_2, A^2\vec{c} = \vec{d}_3$.

Then $\vec{u}(t) = \left(I + At + A^2 t^2\right)\vec{c}$. This answer is slightly different than what was obtained by linear cascades, and it takes more time to produce.

Does the Eigenanalysis method apply? The roots of the characteristic equation are $0, 0, 0$, and the corresponding atoms are $1, x, x^2$. But $A - 0I$ is just A, which has rank 2 and nullity 1. There is only one eigenpair: A is not diagonalizable. The eigenanalysis method *does not apply*.

Laplace theory? Yes, it always works. Begin with the Laplace resolvent equation $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) = \vec{c}$. Solve as $\mathcal{L}(\vec{u}) = (sI - A)^{-1}\vec{c}$. Using the backward Laplace table on each entry in $(sI - A)^{-1}$ gives the answer $\vec{u}(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. None of this is easy, and the preferred method is linear cascades.

[20%] **Ch7(c)**: Find the 2×2 matrix A, given the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Answer: Let $P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, $D = \mathbf{diag}(3, 0)$. Then $A = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

[15%] **Ch7(alt)**: A 2×2 real matrix A has eigenvalues -1 and -2. Display the form of the general solution of the differential equation $\vec{u}' = A\vec{u}$.

Answer: Solution 1. By Cayley-Hamilton-Ziebur the solution is $\vec{u}(t) = \vec{d}_1 e^{-t} + \vec{d}_2 e^{-2t}$ where vectors $\vec{d}_1, \vec{d}_2$ are determined by A and $\vec{u}(0)$.

Solution 2. Distinct eigenvalues implies A is diagonalizable with eigenpairs $(-1, \vec{v}_1), (-2, \vec{v}_2)$. Then for arbitrary constants c_1, c_2 the solution is given by the equation $\vec{u}(t) = c_1 e^{-t}\vec{v}_1 + c_2 e^{-2t}\vec{v}_2$.

[25%] **Ch7(alt)**: Give an example of a 2×2 real matrix A for which Fourier's model is not valid. Then display the general solution $\vec{x}(t)$ of $\vec{x}' = A\vec{x}$.

Answer: Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\lambda = 0, 0$ are the eigenvalues and there is only one eigenvector (a previously worked example). So Fourier's model does not hold: A is not diagonalizable. The general solution is by linear cascades: $x(t) = c_1 t + c_2, y(t) = c_1$.

[25%] **Ch7(alt)**: Consider a 2×2 system $\vec{x}' = A\vec{x}$. Assume A has complex eigenvalues $\lambda = \pm\sqrt{3}i$. Prove that $\lim_{t\to\infty} |\vec{x}(t)| = \infty$ is false for every possible solution $\vec{x}(t)$.

Answer: By Cayley-Hamilton-Ziebur, the solution is a vector linear combination of the atoms $\cos\sqrt{3}t, \sin\sqrt{3}t$. Such linear combinations cannot in general have a limit at $t = \infty$, because the trigonometric functions fail to have a limit at $t = \infty$.

[25%] **Ch7(alt)**: Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 10$ gallons/minute. Let A empty o B at rate r, and let B empty at rate r. Assume the model brine tank mode

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 10, \quad y(0) = 15. \end{cases}$$

Find the maximum amount of salt ever in tank B for $t \geq 0$.

Answer: The maximum of y is when $y' = 0$. Compute $x(t) = 10e^{-t/10}$, $y(t) = 5e^{-t/5}+10e^{-t/10}$ Then $y' = 0$ when $y = 2x$, which can be solved exactly using logs: $t = -10\ln(2)$. Because it occurs for negative time, then the maximum is at $t = 0$. The maximum value is $y(0) = 15$. The salt in tank B decreases for $t \geq 0$, because fresh water is added, which does not add salt to tank B, but flushes out the salt already present.

[25%] **Ch7(alt)**: A 3×3 real matrix A has all eigenvalues equal to zero and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Find the general solution of the differential equation $\vec{x}' = A\vec{x}$.

[50%] **Ch7(alt)**: Apply the eigenanalysis method to solve the system $\vec{x}' = A\vec{x}$, given

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

Answer: The eigenpairs of A are $(-5, \vec{v}_1), (-3, \vec{v}_2), (-4, \vec{v}_3)$ where

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Then

$$\vec{u}(t) = c_1 e^{-5t}\vec{v}_1 + c_1 e^{-3t}\vec{v}_2 + c_1 e^{-4t}\vec{v}_3.$$

[25%] **Ch7(alt)**: Solve for $x(t)$ in the system below. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + y, \\ y' &= -9x + y. \end{aligned}$$

Answer: Let $A = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}$. The roots of $|A - rI| = r^2 - 2r + 10 = 0$ are $1 \pm 3i$. Then by Cayley-Hamilton-Ziebur, $x(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t$. The first differential equation implies $y(t) = x' - x = x + e^t(-3c_1 \sin 3t + 3c_2 \cos 3t) - x = -3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t$.

[25%] **Ch7(alt)**: Consider a 4×4 system $\vec{x}' = A\vec{x}$. Assume A has an eigenvalue $\lambda = -1/7$ with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find a nonzero solution of the differential equation with limit zero at infinity.

Answer: Choose $\vec{x}(t)$ equal to $e^{\lambda t}\vec{v}$, where $\vec{v}$ is one of the two listed eigenvectors.

[25%] **Ch7(alt)**: Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 5, \quad y(0) = 10. \end{cases}$$

Find an equation for the amount of salt in tank B.

Answer: The problem is a linear cascade, to be solved by first order scalar methods. First find $x(t)$ = constant / integrating factor = $5e^{-t/10}$. Substitute this formula into the second equation to get $y' + (1/20)y = (1/2)e^{-t/10}$. Solve for $y = 20e^{-t/20} - 10e^{-t/10}$ by the linear integrating factor method.

Place this page on top of all Ch7 work.

Name _______________ **7:30am**

Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Ch8. (Matrix Exponential) Complete all problems.

[40%] **Ch8(a)**: Using any method in the lectures or the textbook, display the matrix exponential e^{Bt}, for

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Answer: We find a fundamental matrix $Z(t)$. It is done by solving the matrix system $\vec{u}' = B\vec{u}$ with Ziebur's shortcut. Then $x = c_1 \cos 2t + c_2 \sin 2t$, $2y = x' = -2c_1 \sin 2t + 2c_2 \cos 2t$, and finally $y = -c_1 \sin 2t + c_2 \cos 2t$. Write $\vec{u} = Z(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $Z(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}$. Because $Z(0)$ is the identity, then

$$e^{Bt} = Z(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}.$$

Putzer's formula can be used, $e^{Bt} = \text{Real part}\left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(B - \lambda_1 I)\right)$. The eigenvalues $\lambda_1 = 2i$, $\lambda_2 = -2i$ are found from $\det(B - \lambda I) = \lambda^2 + 4 = 0$. A third method would use the formula $e^{Bt} = \mathcal{L}^{-1}((sI - B)^{-1})$ to find the answer in a series of Laplace steps.

[30%] **Ch8(b)**: Consider the 2×2 system

$$\begin{aligned} x' &= 3x, \\ y' &= -y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system as a matrix problem $\vec{u}' = A\vec{u}$ for $\vec{u}$, using the matrix exponential e^{At}.

Answer: Let $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$. The answer for e^{At} can be obtained quickly from the theorem $e^{\mathbf{diag}(a,b)t} = \mathbf{diag}(e^{at}, e^{bt})$, giving the answer $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$. Then use $\vec{u}(t) = e^{At}\vec{u}(0)$, which implies $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{-t} \end{pmatrix}$.

[30%] **Ch8(c)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= 3x + 3, \\ y' &= -y + 1. \end{aligned}$$

Answer: Variation of parameters is $\vec{u}_p(t) = e^{At} \int_0^t e^{-As} \begin{pmatrix} 3 \\ 1 \end{pmatrix} ds$. Then $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$ from the previous problem. Substitute $t \to -s$ to obtain $e^{-As} = \mathbf{diag}(e^{-3s}, e^{s})$. The integration step is

$$\int_0^t \mathbf{diag}(e^{-3s}, e^s) \begin{pmatrix} 3 \\ 1 \end{pmatrix} ds = \int_0^t \begin{pmatrix} 3e^{-3s} \\ e^s \end{pmatrix} ds = \begin{pmatrix} 1 - e^{-3t} \\ e^t - 1 \end{pmatrix}.$$

Finally, $\vec{u}_p(t) = \mathbf{diag}(e^{3t}, e^{-t}) \begin{pmatrix} 1 - e^{-3t} \\ e^t - 1 \end{pmatrix} = \begin{pmatrix} e^{3t} - 1 \\ 1 - e^{-t} \end{pmatrix}$.

Alternate problems.

[30%] **Ch8(alt)**: Check the correct statements.

☐ **1**. The general solution of $\vec{u}' = A\vec{u}$ is a vector linear combination of atoms found from the roots of $\det(A - \lambda I) = 0$.

☐ **2**. The system $\vec{u}' = A\vec{u}$ can only be solved when A is diagonalizable.

☐ **3**. The general solution of $\vec{u}' = A\vec{u}$ can be written as $\vec{u}(t) = e^{At}\vec{u}(0)$.

☐ **4**. The matrix exponential e^{At} can be found using Laplace theory.

☐ **5**. For any $n \times n$ matrix A, $(sI - A)^{-1}$ equals the Laplace integral of e^{At}.

☐ **6**. A second order system $\vec{x}'' = A\vec{x} + \vec{G}(t)$ can be transformed into a first order system of the form $\vec{u}' = B\vec{u} + \vec{F}(t)$.

Answer: The second is false. The others are true.

Place this page on top of all Ch8 work.

Name ______________ **7:30am**

Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

Answer: The eigenvalues of A are roots of $r^2 - 2r + 5 = (r-1)^2 + 4 = 0$, which are complex conjugate roots $1 \pm 2i$. The atoms are exponentials times cosines and sines. Rotation eliminates the saddle and node. Finally, the atoms $e^t \cos 2t$, $e^t \sin 2t$ have limit zero at $t = -\infty$, therefore the system is unstable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 - y + 9, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 3$, $y = 3$. Compute the Jacobian matrix $A = J(3,3)$ of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -2y-1 \\ 4x-2y & -2x \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} 1 & -7 \\ 6 & -6 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x + 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{aligned}$$

At equilibrium point $x = -3$, $y = 3$, the Jacobian matrix is $A = J(-3,3) = \begin{pmatrix} 10 & 4 \\ 3 & 3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\vec{u}' = A\vec{u}$.

(2) Apply a theorem to classify $x = -3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 4-2x & 4 \\ 3 & 3 \end{pmatrix}$. Then $A = J(-3,3) = \begin{pmatrix} 10 & 4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 - 13r + 18 = 0$, giving two different positive roots a, b. Because both atoms e^{at}, e^{bt} are real, rotation does not happen, and the classification must be a saddle or a node. The atoms have a limit at $t = -\infty$, therefore it is a node, and we report an **unstable node** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$. Theorem 2 in 9.2 applies to say that the same is true for the nonlinear system: **unstable node** at $x = -3$, $y = 3$.

Alternate problems

[10%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x + y, \\ y' &= x + y + 4 - x^2. \end{aligned}$$

Find the equilibrium points of the nonlinear system.

Answer: None supplied.

Place this page on top of all Ch9 work.

Name ______________ **7:30am**

Mathematics 2250-1 Sample Final Exam Problems at 7:30am on 27 Apr 2012

Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{1}{s^3}$	$\dfrac{1}{s-4}$	$\dfrac{s}{s^2+4}$	$\dfrac{2}{s^2+1}$	$\dfrac{e^{-s}}{s+1}$

$f(t)$	t	te^t	$t\cos 2t$	$t(1+e^t)$	$e^t \sin t$
$\mathcal{L}(f(t))$					

Answer: First table left to right: $\dfrac{t^2}{2}$, e^{4t}, $\cos 2t$, $2\sin t$, $e^{1-t}\,\mathbf{step}(t-1)$. Function $\mathbf{step}(t)$ is the unit step $u(t)$ of the textbook, $\mathbf{step}(t) = 1$ for $t \ge 0$, zero elsewhere. Second table left to right: $\dfrac{1}{s^2}$, $\dfrac{1}{(s-1)^2}$, $\dfrac{-d}{ds}\dfrac{s}{s^2+4} = \dfrac{s^2-4}{(s^2+4)^2}$, $\dfrac{1}{s^2} + \dfrac{1}{(s-1)^2}$, $\dfrac{1}{(s-1)^2+1}$.

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for the pulse $f(t) = t$ on $1 \le t < 2$, $f(t) = 0$ otherwise.

Answer: Define $\mathbf{step}(t)$ to be the unit step. Use $f(t) = t\,\mathbf{step}(t-1) - t\,\mathbf{step}(t-2)$, and the second shifting theorem. Then $\mathcal{L}(t\,\mathbf{step}(t-1)) = e^{-s}\mathcal{L}(t|_{t\to t+1}) = e^{-s}\mathcal{L}(t+1) = e^{-s}(\frac{1}{s^2}+\frac{1}{s})$. Similarly, $\mathcal{L}(t\,\mathbf{step}(t-2)) = e^{-2s}\mathcal{L}(t|_{t\to t+2}) = e^{-2s}\mathcal{L}(t+2) = e^{-2s}(\frac{1}{s^2}+\frac{2}{s})$. The answer: $\mathcal{L}(f(t)) = e^{-s}(\frac{1}{s^2}+\frac{1}{s}) - e^{-2s}(\frac{1}{s^2}+\frac{2}{s})$.

[30%] **Ch10(c)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - 2x'(t) = 4e^{2t}, \quad x(0) = x'(0) = 0.$$

Answer: $x(t) = 1 - e^{2t} + 2te^{2t}$. The Laplace steps are: (1) $\mathcal{L}(x(t)) = \dfrac{4}{s(s-2)^2}$; (2) $\mathcal{L}(x(t)) = \dfrac{4}{s(s-2)^2} = \dfrac{A}{s} + \dfrac{B}{s-2} + \dfrac{C}{(s-2)^2} = \mathcal{L}(A + Be^{2t} + Cte^{2t})$ where by partial fractions the answers are $A = 1$, $B = -1$, $C = 2$. Lerch's theorem implies $x(t) = 1 - e^{2t} + 2te^{2t}$.

Alternate problems.

[20%] **Ch10(alt)**: Let $f(t) = |\sin t|$ on $0 \le t \le 2\pi$, with $f(t)$ periodic of period 2π. Display the formula for $\mathcal{L}(f(t))$ according to the periodic function rule. To save time, **don't evaluate any integrals**.

[20%] **Ch10(alt)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-s}}{s-2}$.

Answer: Use the second shifting theorem, $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)\,\mathbf{step}(t-a))$. Then $\mathcal{L}(f(t)) = e^{-s}\mathcal{L}(e^{-2t}) = \mathcal{L}(\left.e^{-2u}\right|_{u=t-1}\mathbf{step}(t-1)) = \mathcal{L}(e^{2-2t}\,\mathbf{step}(t-1))$. Lerch's theorem implies $f(t) = e^{2-2t}\,\mathbf{step}(t-1)$.

[20%] **Ch10(alt)**: Let $f(t) = |\sin t|$ on $0 \le t \le 2\pi$, with $f(t)$ periodic of period 2π. Display the formula for $\mathcal{L}(f(t))$ according to the periodic function rule. To save time, **don't evaluate any integrals**.

Answer: The formula is $\mathcal{L}(f(t)) = \frac{\int_0^{2\pi} |\sin(t)|e^{-st}dt}{1-e^{-2\pi s}}$.

[20%] **Ch10(alt)**: Solve for $f(t)$ in the equation $\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{1}{(s+1)^2} + \dfrac{d^2}{ds^2}\mathcal{L}(\sin t)$.

Answer: $\mathcal{L}((-t)f(t)) = \left.\frac{1}{u^2}\right|_{u=s+1} + \mathcal{L}((-t)^2 \sin t) = \mathcal{L}(e^{-t}\,(t) + t^2 \sin t)$, giving the final answer $f(t) = -e^{-t} - t\sin t$. The Laplace details use the first shifting theorem and s-differentiation theorem.

[40%] **Ch10(alt)** Use Laplace's method to find an explicit formula for $x(t)$. Don't find $y(t)$!

$$\begin{array}{rcrcl} x'(t) &=& 2x(t) &+& 5y(t), \\ y'(t) &=& 5x(t) &+& 2y(t), \\ x(0) &=& 1, && \\ y(0) &=& 1. && \end{array}$$

Answer: Find the resolvent equation $L(\vec{u}) = (sI - A)^{-1}\vec{u}(0)$. After some details, $x = y = e^{7t}$.

Additional Laplace problems without answers

[25%] **Ch10(alt)**: Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

[15%] **Ch10(alt)**: Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\left.\mathcal{L}\left(t^2 e^{3t}\right)\right|_{s \to (s+3)}\right).$$

[20%] **Ch10(alt)**: Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

[15%] **Ch10(alt)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

[25%] **Ch10(alt)**: Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\dfrac{\sin(t)}{t}$.

[20%] **Ch10(alt)**: Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{s+2}$	$\dfrac{s+1}{s^2+2s+5}$		

[30%] **Ch10(alt)**: Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

[35%] **Ch10(alt)**: Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. To save time, **do not** solve for $x(t)$ or $y(t)$!

$$\begin{aligned} &x'' = 3x + 3y + 2, \\ &y'' = 4x + 2y, \\ &x(0) = 0, \quad x'(0) = 2, \\ &y(0) = 0, \quad y'(0) = 3. \end{aligned}$$

[35%] **Ch10(alt)**: Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t+\sin t)\big|_{s\to(s-2)}.$$

[30%] **Ch10(alt)**: Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

[30%] **Ch10(alt)**: Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. To save time, **do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} - x'' = 3t^2 + 4e^{-2t} + 5e^t\sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

Place this page on top of all Ch10 work.