

Name KEY ver 2

Differential Equations and Linear Algebra 2250
 Midterm Exam 2
 Version 2, 22 Mar 2012

Scores
1.
2.
3.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (The 3 Possibilities with Symbols)

Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} c-a & -1 & 0 \\ a-c & 1 & c \\ 4c-2a & 3 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ b^2-b \\ b^2 \end{pmatrix}$$

- (a) [40%] Determine a, b and c such that the system has a unique solution.
- (b) [30%] Explain why $c = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.
- (c) [30%] Explain why $c = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

(a) System $A\vec{x} = \vec{B}$ has a unique solution $\vec{x} \Leftrightarrow |A| \neq 0$.

$$\left(\begin{array}{ccc|c} c-a & -1 & 0 & b \\ a-c & 1 & c & b^2-b \\ 4c-2a & 3 & c & b^2 \end{array} \right) \begin{array}{l} \text{combo}(1,2,1) \\ \text{Combo}(1,3,3) \end{array} \rightarrow \left(\begin{array}{ccc|c} c-a & -1 & 0 & b \\ 0 & 0 & c & b^2 \\ 7c-5a & 0 & c & b^2+3b \end{array} \right)$$

$$|A| = (-1)(-1) \begin{vmatrix} c & c \\ 7c-5a & c \end{vmatrix} = -c(7c-5a). \quad \boxed{\text{Unique solution} \Leftrightarrow c(7c-5a) \neq 0}$$

(b) If $c=0$ and $b \neq 0$, then the last frame is

$$\left(\begin{array}{ccc|c} -a & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \\ -5a & 0 & 0 & b^2+3b \end{array} \right). \text{ The second eq is } 0 = b^2, \text{ a signal equation.}$$

The system has no solution.

(c) If $c=b=0$, then the frame above in (b) becomes

$$\left(\begin{array}{ccc|c} -a & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5a & 0 & 0 & 0 \end{array} \right). \text{ This system has solution } \vec{x} = \vec{0}, \text{ so it is consistent. Because } x_3 \text{ does not appear, it is a free variable. Therefore, the system has } \infty \text{-many solutions.}$$

Use this page to start your solution. Attach extra pages as needed.

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2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

- (a) [10%] True or false: There is no subspace S of \mathcal{R}^3 containing all of the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
 $S = \mathcal{R}^3$ is a subspace containing all of the vectors.
- (b) [10%] True or false: The set of solutions \vec{x} in \mathcal{R}^3 of a consistent matrix equation $A\vec{x} = \vec{b}$ can equal just the zero vector. Let A be the identity matrix and let $\vec{b} = \vec{0}$.
- (c) [10%] True or false: Relations $2xy = 0, x + z = 0$ define a subspace in \mathcal{R}^3 . See below
- (d) [10%] True or false: Equations $x + y = 0, x + y + z = 1$ define a subspace in \mathcal{R}^3 . See below
- (e) [20%] Two linear algebra theorems are able to conclude that the set S of all linear combinations of the functions $\sin(2x), e^x, \cosh(2x)$ is a vector space of functions. State the two theorems.
- (f) [40%] Find a basis of vectors for the subspace of \mathcal{R}^4 given by the system of restriction equations

$$\begin{aligned} 3x_1 + 10x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + x_4 &= 0 \\ -2x_1 + 4x_2 &= 0 \\ 2x_1 + 12x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

(c) use the not a subspace theorem. vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ satisfy the relations but their sum $= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ does not, so it's not a subspace.

(d) The zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ does not satisfy $x+y+z=1$ (get $0=1$) so it's not a subspace, by the not a subspace theorem. or, use the subspace criterion (Thm 1 in 4.1): item 1 fails, that $\vec{0}$ is in S .

- (e) • Theorem (span Thm) The set S of all linear combinations of a set of vectors in vector space V is a subspace of V .
- Theorem (subspace criterion). A set S in vector space V is a subspace provided 1, 2, 3 hold:
 1. $\vec{0}$ is in S ; 2. \vec{x}, \vec{y} in $S \Rightarrow \vec{x} + \vec{y}$ in S ; 3. \vec{x} in $S, c = \text{scalar} \Rightarrow c\vec{x}$ in S .

(f) The augmented matrix is

$$\left(\begin{array}{cccc|c} 3 & 10 & 2 & 2 & 0 \\ 2 & 4 & 1 & 1 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & 12 & 2 & 2 & 0 \end{array} \right). \text{ Its rref} = \left(\begin{array}{cccc|c} 1 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1 & 1/8 & 1/8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

$$\text{Then } \begin{cases} x_1 + \frac{1}{4}x_3 + \frac{1}{4}x_4 = 0 \\ x_2 + \frac{1}{8}x_3 + \frac{1}{8}x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \text{ implies } \vec{x} = \begin{pmatrix} -\frac{1}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{1}{8}x_3 - \frac{1}{8}x_4 \\ x_3 \\ x_4 \end{pmatrix}. \text{ Basis} = \frac{\partial \vec{x}}{\partial x_3}, \frac{\partial \vec{x}}{\partial x_4}$$

Use this page to start your solution. Attach extra pages as needed.

Subspace = Span(Basis)

$$\text{Basis} = \left\{ \begin{pmatrix} -1/4 \\ -1/8 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/4 \\ -1/8 \\ 0 \\ 1 \end{pmatrix} \right\}. \text{ Subspace} = \text{Span} \left(\begin{pmatrix} -1/4 \\ -1/8 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/4 \\ -1/8 \\ 0 \\ 1 \end{pmatrix} \right).$$

