Name KEY ver 2

Scores
1.
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3.

Differential Equations and Linear Algebra 2250

Midterm Exam 2

Version 2, 22 Mar 2012

Instructions This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} c-a & -1 & 0 \\ a-c & 1 & c \\ 4c-2a & 3 & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ b^2-b \\ b^2 \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution.

(b) [30%] Explain why $c = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why $c = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

(a) System $A\vec{x} = \vec{B}$ has a unique solution $\vec{x} \iff |A| \neq 0$.

$$\left(\begin{array}{ccc|c} c-a & -1 & 0 & b \\ a-c & 1 & c & b^2-b \\ 4c-2a & 3 & c & b^2 \end{array}\right) \begin{array}{c} \text{combo}(1,2,1) \\ \text{combo}(1,3,3) \end{array} \longrightarrow \left(\begin{array}{ccc|c} c-a & -1 & 0 & b \\ 0 & 0 & c & b^2 \\ 7c-5a & 0 & c & b^2+3b \end{array}\right)$$

$$|A| = (-1)(-1)\begin{vmatrix} 0 & c \\ 7c-5a & c \end{vmatrix} = -c(7c-5a).$$ $\boxed{\text{Unique solution} \iff c(7c-5a) \neq 0}$

(b) If $c=0$ and $b \neq 0$, Then The last frame is

$$\left(\begin{array}{ccc|c} -a & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \\ -5a & 0 & 0 & b^2+3b \end{array}\right).$$ The second eq is $0 = b^2$, a signal equation. The system has no Solution.

(c) If $c=b=0$, Then The frame above in (b) becomes

$$\left(\begin{array}{ccc|c} -a & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5a & 0 & 0 & 0 \end{array}\right).$$ This system has solution $\vec{x} = \vec{0}$, so it is Consistent. Because x_3 does not appear, it is a free variable. Therefore, The system has ∞-many solutions.

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2 **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is no subspace S of $\mathcal{R}^3$ containing all of the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$

$S = \mathbb{R}^3$ is a subspace containing all of the vectors.

(b) [10%] True or false: The set of solutions $\vec{x}$ in $\mathcal{R}^3$ of a consistent matrix equation $A\vec{x} = \vec{b}$ can equal just the zero vector. Let A be the identity matrix and let $\vec{b} = \vec{0}$.

(c) [10%] True or false: Relations $2xy = 0$, $x + z = 0$ define a subspace in $\mathcal{R}^3$. See below

(d) [10%] True or false: Equations $x + y = 0$, $x + y + z = 1$ define a subspace in $\mathcal{R}^3$. See below

(e) [20%] Two linear algebra theorems are able to conclude that the set S of all linear combinations of the functions $\sin(2x)$, e^x, $\cosh(2x)$ is a vector space of functions. State the the two theorems.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 10x_2 & + & 2x_3 & + & 2x_4 & = & 0, \\ 2x_1 & + & 4x_2 & + & x_3 & + & x_4 & = & 0, \\ -2x_1 & + & 4x_2 & & & & & = & 0, \\ 2x_1 & + & 12x_2 & + & 2x_3 & + & 2x_4 & = & 0. \end{array}$$

(c) use The not a subspace Theorem. Vectors $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ satisfy The relations but Their sum $= \begin{pmatrix}1\\1\\-1\end{pmatrix}$ does not, so its not a subspace.

(d) The zero vector $\begin{pmatrix}0\\0\\0\end{pmatrix}$ does not satisfy $x+y+z=1$ (get $0=1$) so its not a subspace, by The not a subspace Theorem. Or, use The subspace criterion (Thm 1 in 4.1): item 1 fails, That $\vec{0}$ is in S.

(e) • Theorem (Span Thm) The set S of all linear combinations of a set of vectors in vector space V is a subspace of V.

• Theorem (subspace criterion). A set S in vector space V is a subspace provided 1, 2, 3 hold:

1. $\vec{0}$ is in S; 2. $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x}+\vec{y}$ in S; 3. $\vec{x}$ in S, c = scalar $\Rightarrow c\vec{x}$ in S.

(f) The augmented matrix is

$$\left(\begin{array}{cccc|c} 3 & 10 & 2 & 2 & 0 \\ 2 & 4 & 1 & 1 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & 12 & 2 & 2 & 0 \end{array}\right) \quad \text{Its rref} = \left(\begin{array}{cccc|c} 1 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1 & 1/8 & 1/8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Then $\begin{cases} x_1 + \frac{1}{4}x_3 + \frac{1}{4}x_4 = 0 \\ x_2 + \frac{1}{8}x_3 + \frac{1}{8}x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$ implies $\vec{x} = \begin{pmatrix} -\frac{1}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{1}{8}x_3 - \frac{1}{8}x_4 \\ x_3 \\ x_4 \end{pmatrix}$. Basis $= \frac{\partial \vec{x}}{\partial x_3}, \frac{\partial \vec{x}}{\partial x_4}$

Subspace = Span(Basis)

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Basis $= \begin{pmatrix}-1/4\\-1/8\\1\\0\end{pmatrix}, \begin{pmatrix}-1/4\\-1/8\\0\\1\end{pmatrix}$. Subspace = Span$\left(\begin{pmatrix}-1/4\\-1/8\\1\\0\end{pmatrix}, \begin{pmatrix}-1/4\\-1/8\\0\\1\end{pmatrix}\right)$.

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3 **(Independence and Dependence)** Do **all parts.**

(a) [10%] State an independence test for 4 vectors in $\mathcal{R}^4$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State another [different than (a)] independence test for 4 vectors in $\mathcal{R}^4$.

(c) [10%] For any matrix A, $\mathbf{rank}(A)$ equals the number of lead variables for the problem $A\vec{x} = \vec{0}$. How many non-pivot columns in a 10×10 matrix A with $\mathbf{rank}(A) = 7$?

(d) [30%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & -6 & 0 \\ 0 & 2 & 0 & 5 & 1 \end{pmatrix}.$$

Decide if the four rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are independent and display the details of the chosen independence test.

(e) [40%] Extract from the list below a largest set of independent vectors.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_5 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \\ 9 \end{pmatrix}, \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

(a) The 4 vectors in $\mathbb{R}^4$ are independent $\Leftrightarrow$ rank of the augmented matrix of the 4 vectors equals 4.

(b) Four vectors in $\mathbb{R}^4$ are independent $\Leftrightarrow$ the determinant of the augmented matrix of the four vectors is nonzero.

(c) The number of pivots = rank = 7. There are 10 columns, so there are 3 non-pivot cols.

(d) Add rows 1, 2 to obtain row 4. The rows are dependent.
<u>Alternate</u>: Compute $|A| = 0$, then apply the determinant test stated in (b) above to get $|A^T| = |A| = 0 \Rightarrow$ cols of A^T are dependent $\Rightarrow$ rows of A are dependent.

(e) Form the augmented matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & -1 & -2 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 1 & 9 & 2 \end{pmatrix}. \text{ Then rref} = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & \frac{-3}{2} \\ & & \text{4 zero rows} & & & \end{pmatrix}.$$

pivots = 2, 3.

Largest indep subset $= \{\vec{v}_2, \vec{v}_3\}$.

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