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## Three Examples

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Growth-Decay  $\frac{dA}{dt} = kA(t),$   
 $A(0) = A_0$

$$A(t) = A_0 e^{kt}$$

Newton Cooling  $\frac{du}{dt} = -h(u(t) - u_1),$   
 $u(0) = u_0$

$$u(t) = u_1 + (u_0 - u_1)e^{-ht}$$

Verhulst Logistic  $\frac{dP}{dt} = (a - bP(t))P(t),$   
 $P(0) = P_0$

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

### Theorem 1 (Peano)

Let  $f(x, y)$  be continuous in a domain  $D$  of the  $xy$ -plane and let  $(x_0, y_0)$  belong to the interior of  $D$ . Then there is a small  $h > 0$  and a function  $y(x)$  continuously differentiable on  $|x - x_0| < h$  such that  $(x, y(x))$  remains in  $D$  for  $|x - x_0| < h$  and  $y(x)$  is one solution (many more might exist) of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

### Meaning

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Given a box

$$B = \{(x, y) : |x - x_0| \leq H, |y - y_0| \leq b\}$$

with center  $(x_0, y_0)$  and  $f$  continuous on  $B$ , then there is smaller box

$$B_1 = \{(x, y) : |x - x_0| \leq h, |y - y_0| \leq b\}$$

which contains an edge-to-edge solution curve  $y(x)$  of  $y' = f(x, y)$  passing through the center of the box.

### Definition 1 (Picard Iteration)

Define the constant function  $y_0(x) = y_0$  and then define by iteration

$$y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t)) dt.$$

The sequence  $y_0(x), y_1(x), \dots$  is called the **sequence of Picard iterates** for  $y' = f(x, y), y(x_0) = y_0$ .

**Growth-Decay Example**  $y' = ky, y(0) = y_0$  \_\_\_\_\_

The first few iterates are

$$y_0(x) = y_0 \quad (\text{a constant function})$$

$$y_1(x) = y_0 + kxy_0$$

$$y_2(x) = y_0 + kxy_0 + (kx)^2 y_0 / 2$$

$$y_3(x) = y_0 + kxy_0 + (kx)^2 y_0 / 2 + (kx)^3 y_0 / 6.$$

## Theorem 2 (Picard-Lindelöf)

Let  $f(x, y)$  and  $f_y(x, y)$  be continuous in a domain  $D$  of the  $xy$ -plane. Let  $(x_0, y_0)$  belong to the interior of  $D$ . Then there is a small  $h > 0$  and a *unique* function  $y(x)$  continuously differentiable on  $|x - x_0| < h$  such that  $(x, y(x))$  remains in  $D$  for  $|x - x_0| < h$  and  $y(x)$  solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n \rightarrow \infty} y_n(x) = y(x)$$

is satisfied for  $|x - x_0| < h$  by the Picard iterates  $y_n$ .

### Explanation

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Similar to the Peano Theorem, given a box  $B$  on which  $f$  and  $f_y$  are continuous, there is a smaller box  $B_1$  and *exactly one* edge-to-edge solution curve  $y(x)$  of  $y' = f(x, y)$  passing through the center of the box.

### Engineering translation

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If  $f$  and  $f_y$  are continuous, then standard numerical methods apply to approximate the unique solution  $y(x)$ .

**1 Example (Applying Picard's Theorem)** Consider the problem  $y' = 3(y - 1)^{2/3}$ ,  $y(0) = 1$ . Explain why Peano's theorem applies but Picard's theorem does not.

### Solution

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The initial value problem is written as  $y' = f(x, y)$ ,  $y(x_0) = y_0$ . Then  $x_0 = 0$ ,  $y_0 = 1$  and

$$f(x, y) = 3(y - 1)^{2/3}, \quad f_y(x, y) = 2(y - 1)^{-1/3}.$$

Any selected box  $B$  with center  $x = 0$ ,  $y = 1$  satisfies  $f$  continuous on  $B$ . Peano's theorem applies to conclude the initial value problem has an edge-to-edge solution in a smaller box  $B_1$ .

There is no such box  $B$  on which  $f_y$  is continuous, because  $f_y$  is undefined at  $y = 1$ . Picard's theorem does not apply.

### Conclusion

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We cannot conclude there are multiple solutions, based only on Picard's theorem. In order to conclude there are two or more solutions, we have to exhibit them. It turns out that  $y = 1$  and  $y = 1 + x^3$  are two solutions. Sometimes, the only signal for multiple solutions is failure of the Picard theorem hypothesis.



## What do the arrows represent?

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The re-scaled vector is the tangent line direction for the solution  $\mathbf{y}(x)$  of the initial value problem

$$\begin{cases} \mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \\ \mathbf{y}(x_0) = \mathbf{y}_0. \end{cases}$$

The curve  $\mathbf{y}(x)$  has position vector  $\vec{r}(x) = x\vec{i} + \mathbf{y}(x)\vec{j}$ . Its velocity vector at  $(x_0, \mathbf{y}_0)$  is

$$\frac{d\vec{r}}{dx} = \vec{i} + \mathbf{y}'(x_0)\vec{j} = \vec{i} + \mathbf{f}(x_0, \mathbf{y}_0)\vec{j}.$$

## Short solution curves

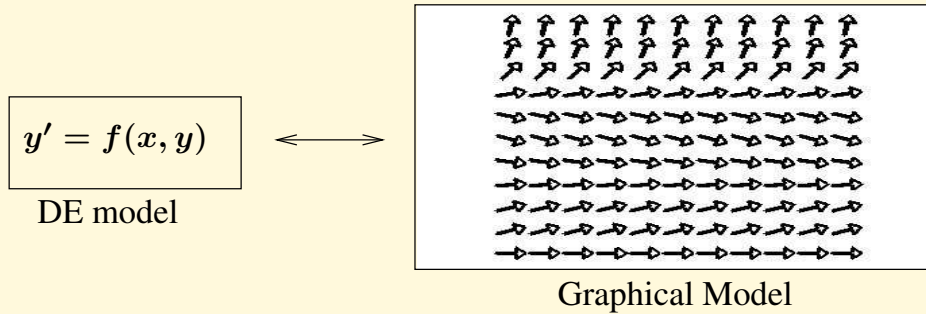
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To pixel resolution, an arrow shaft is a *short solution curve* to  $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ , because the tangent line approximates the curve at the contact point.



## Model Replacement

Euler's idea replaces the differential equation model  $y' = f(x, y)$  by a graphical model, a direction field.



**Figure 2.** Model Replacement.

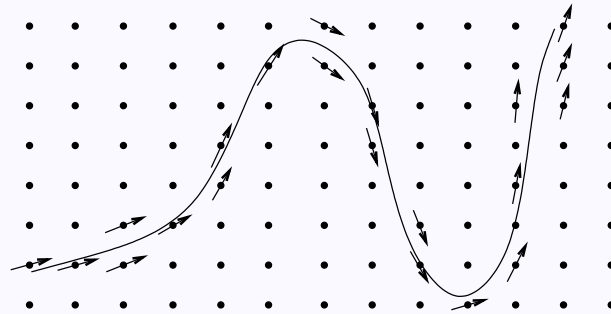
A differential equation model  $y' = f(x, y)$  is replaced by a direction field model, a graphic consisting of grid points and line segments. A line segment represents a solution curve through the grid point, to pixel resolution. Model replacement means the differential equation model is tossed aside and we deal only with the graphical model.

## Rules for Drawing Threaded Solutions

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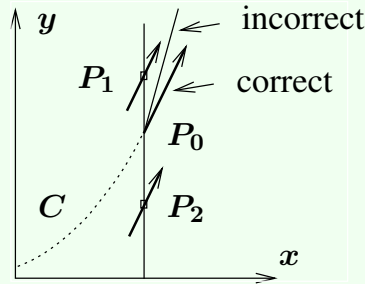
A direction field graphic replaces all the information supplied by the equation  $y' = f(x, y)$ . The equation is tossed aside and not used. *Visualization* of all solutions involves drawing a small number of edge-to-edge solutions  $y(x)$  onto the direction field graph window. Used are just these rules:

1. Solutions don't cross.
2. Nearby tangents nearly match.



**Figure 3.** Threading a Solution Edge-to-Edge.

Shown is a threaded solution curve for  $y' = f(x, y)$  plus nearby grid points and relevant line segments (arrows). The solution threads its way through the direction field, matching tangents at nearby grid points. Technically, the arrows cannot touch the threaded curve, unless the arrow lies atop the curve.



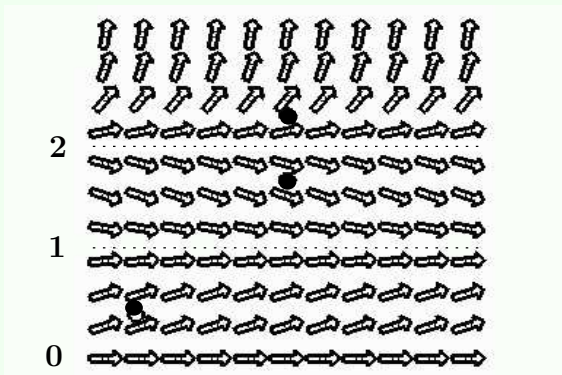
**Figure 4.** Threading Rules.

Solution curve  $C$  threads from the left and meets a line at point  $P_0$ . The line contains two nearby grid points  $P_1, P_2$ . The tangent at  $P_0$  must nearly match direction field arrows at  $P_1, P_2$ .

### Explanation

The slopes of the tangents in Figure 4 are given by  $y' = f(x, y)$ . For points  $P_1 = (x_1, y_1)$ ,  $P_0 = (x_0, y_0)$  and  $P_2 = (x_2, y_2)$ , the slopes are  $f(x_1, y_1)$ ,  $f(x_0, y_0)$ ,  $f(x_2, y_2)$ . If  $P_0, P_1, P_2$  are close, then continuity of  $f$  implies all three slopes are close.

**2 Example (Threading a Solution)** Starting at the black dots in the direction field graphic of Figure 5, thread three solution curves.

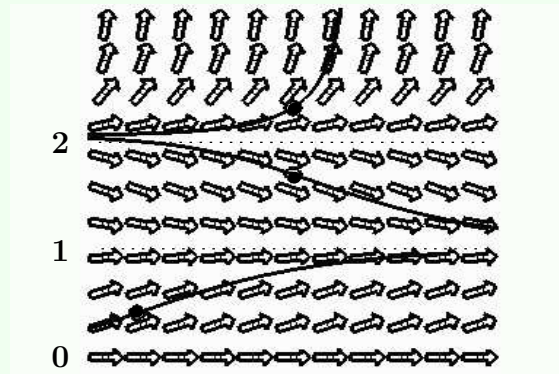


**Figure 5.** A direction field.

A field for the differential equation  $y' = y(2 - y)(1 - y)$  is plotted on graph window  $0 \leq x \leq 3$ ,  $0 \leq y \leq 3$ . The black dots are at  $(0.25, 0.4)$ ,  $(1.5, 2.25)$  and  $(1.5, 1.65)$ .

## Solution

A plot appears in Figure 6. A threaded solution matches its tangents with nearby lineal elements of the direction field in Figure 5. Each threaded curve represents a solution of the differential equation through the given dot on the entire interval  $0 \leq x \leq 3$ , whereas the lineal elements represent solutions through the grid point on a very short  $x$ -interval.



**Figure 6.** Threaded solutions.

The graph window is  $0 \leq x \leq 3$ ,  $0 \leq y \leq 3$ . Threaded curves cannot cross direction field segments nor other threaded solutions, including the constant threaded solutions  $y = 0$ ,  $y = 1$  and  $y = 2$ .