Ch 4 EXERCISES 2250

1-4. P237

Given $a = 2i - j$ and $b = j - 3k$, find $|a-b|$, $2a+b$, $3a-b$.

$|a-b| = |2i - j - (j - 3k)|$
$= |2i - 2j + 3k|$
$= \sqrt{2^2+2^2+3^2}$
$= \sqrt{17}$

$2a+b = 2(2i-j) + j - 3k$
$= 4i - j - 3k$
or $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$

$3a-b = 3(2i-j) - (j-3k)$
$= 6i - 4j + 3k$
or $\begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$

4.1-12 P237

write $w = au + bv$ for some a, b given $u = (4,1)$, $v = (-2,-1)$, $w = (2,-2)$.

To satisfy

$a\begin{bmatrix} 4 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

requires the rref method applied to the augmented matrix.

$\left[\begin{array}{cc|c} 4 & -2 & 2 \\ 1 & -1 & -2 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$ in 4 steps (omitted here)

Then $a = 3$, $b = 5$.

Check:

$au + bv = 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -2 \\ -1 \end{bmatrix}$
$= \begin{bmatrix} 12-10 \\ 3-5 \end{bmatrix}$
$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
$= w.$

4.1-18 P237

Given $u = (1,1,0)$, $v = (4,3,1)$, $w = (3,-2,-4)$ determine dependence or independence by two methods.

Method 1 (rref method)

To solve $au + bv + cw = 0$ for a, b, c, write it as the system

$$\begin{bmatrix} 1 & 4 & 3 \\ 1 & 3 & -2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then apply rref methods:

$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$
$\approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ in 4 steps

Conclude $a = 0, b = 0, c = 0$
$\therefore$ independent.

Method 2 (determinants)

Apply Thm 3.

$\det\begin{bmatrix} 1 & 4 & 3 \\ 1 & 3 & -2 \\ 0 & 1 & -4 \end{bmatrix}$

$= (1)[-12+2] - (1)[-16-3]$
$= 8$
$\neq 0$
$\therefore$ independent.

4.2-16 P245

Find two solution vectors u, v such that $x = su + tv$ is the general solution of the system

$$\begin{cases} x_1 - 4x_2 - 3x_3 - 7x_4 = 0 \\ 2x_1 - x_2 + x_3 + 7x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 = 0 \end{cases}$$ The Given system

$$\left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{array}\right]$$ augmented matrix equivalent.

$$\cong \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 7 & 7 & 21 & 0 \\ 0 & 6 & 6 & 18 & 0 \end{array}\right]$$ add multiples of row 1 to the other rows

$$\cong \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 & 0 \end{array}\right]$$ Divide rows 2,3 to get leading ones

$$\cong \left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cong \left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ rref found

$$\begin{cases} x_1 + x_3 + 5x_4 = 0 \\ x_2 + x_3 + 3x_4 = 0 \end{cases}$$ Equiv system

x_1, x_2 = lead vars

x_3, x_4 = free vars

$$\begin{cases} x_1 = -s - 5t \\ x_2 = -s - 3t \\ x_3 = s \\ x_4 = t \end{cases} \quad \text{or} \quad X = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{u = (-1,-1,1,0) \quad v = (-5,-3,0,1)}$$

4.3-13 P254

Write w as a linear combination of v_1, v_2, if possible, given

$w = (5,2,-2)$, $v_1 = (1,5,-3)$, $v_2 = (5,-$

To solve $w = av_1 + bv_2$ for a, b requires solution of matrix equation

$$\begin{bmatrix} 1 & 5 \\ 5 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$$ or its equivalent augment sy

$$\left[\begin{array}{cc|c} 1 & 5 & 5 \\ 5 & -3 & 2 \\ -3 & 4 & -2 \end{array}\right]$$

$$\cong \begin{bmatrix} 1 & 5 & 5 \\ 2 & 1 & 0 \\ -3 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 5 \\ 2 & 1 & 0 \\ 0 & 10 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 5 \\ 0 & -9 & -10 \\ 0 & 10 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 5 \\ 0 & -9 & -10 \\ 0 & 1 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 5 \\ 0 & 0 & -73 \\ 0 & 1 & -7 \end{bmatrix} \longleftarrow \text{inconsistent!}$$

No solution a, b exists.

4.3-26

Given v_1, v_2, v_3 are independent, show that
$$u_1 = v_2 + v_3,\quad u_2 = v_1 + v_3,\quad u_3 = v_1 + v_2$$
are also independent.

Let c_1, c_2, c_3 be constants and assume
$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0.$$
It will be shown that $c_1 = c_2 = c_3 = 0$, hence u_1, u_2, u_3 are independent.

$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0$ — assumed

$c_1(v_2+v_3) + c_2(v_1+v_3) + c_3(v_1+v_2) = 0$ — subst for u_1, u_2, u_3 given eqtns.

$(c_2+c_3)v_1 + (c_1+c_3)v_2 + (c_1+c_2)v_3 = 0$ — Regroup on v_1, v_2, v_3.

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \end{cases}$$
apply independence of v_1, v_2, v_3.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
system form

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 \neq 0$$

By Cramer's Rule, $c_1 = c_2 = c_3 = 0$.

4.4-20 P262

Find a basis for the solution space of the system
$$\begin{cases} x_1 - 3x_2 - 10x_3 + 5x_4 = 0 \\ x_1 + 4x_2 + 11x_3 - 2x_4 = 0 \\ x_1 + 3x_2 + 8x_3 - x_4 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{pmatrix}$$
$$\cong \begin{pmatrix} 1 & -3 & -10 & 5 \\ 0 & 7 & 21 & -7 \\ 0 & 6 & 18 & -6 \end{pmatrix}$$
$$\cong \begin{pmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$
$$\cong \begin{pmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\cong \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
rref found

lead vars: x_1, x_2

free vars: x_3, x_4

$$\begin{cases} x_1 = s - 2t \\ x_2 = -3s + t \\ x_3 = s \\ x_4 = t \end{cases}$$

$$X = s\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\boxed{(1,-3,1,0),\ (-2,1,0,1)}$

Check the answer

$$\begin{pmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+ \\ 1- \\ 1\cdot \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ - \\ - \end{pmatrix}$$
$$= ($$

The answer works!

4.5-17, p270 Determine independence or dependence for $\cos 2x$, $\sin^2 x$, $\cos^2 x$.

Dependent. Apply a trig identity to find a, b, c not all zero such that $a \cos 2x + b \sin^2 x + c \cos$ for all x.

4.5-22 p 270 Solve for A, B, C given

$$\frac{2x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Clear fractions by multiplication across by the product $(x+1)(x+2)(x+3)$. Then

$$2x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Set successively $x=-1$, $x=-2$, $x=-3$ (roots of the denominator on the left). Then

$$\begin{cases} -2 = A(1)(2) + B(0) + C(0) \\ -4 = A(0) + B(-1)(1) + C(0) \\ -6 = A(0) + B(0) + C(-2)(-1) \end{cases}$$

Then $\boxed{A=-1, B=4, C=-3}$

Check: multiply across by $x+1$ and set $x+1=0$ to get $\frac{-2}{(1)(2)} = A$. Repeat for $x+2$ and $x+3$. This is called Heaviside's Covering method.