

Separable Differential Equations

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Definition (Separable Equation). An equation $y' = f(x, y)$ is called **separable** provided there exists functions $F(x)$ and $G(y)$ such that

$$f(x, y) = F(x)G(y).$$

Definition (Separated Form of a Separable Equation). The equation

$$\frac{y'}{G(y)} = F(x).$$

is called the **separated form**. It is obtained from the separable equation $y' = F(x)G(y)$ by dividing by $G(y)$.

Such an equation is said to be *prepared for quadrature*, because the left side is independent of x and the right side is independent of y, y' .

Finding a Separable Form

The algorithm supplied here determines F and G such that $f(x, y) = F(x)G(y)$. The algorithm also applies to **prove** that an equation is **not separable**.

Algorithm. Given differential equation $y' = f(x, y)$, invent values x_0, y_0 such that $f(x_0, y_0) \neq 0$. Define F, G by the formulas

$$(1) \quad F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \quad G(y) = f(x_0, y).$$

Because $f(x_0, y_0) \neq 0$, then (1) makes sense. Test *infra* implies the following test.

Theorem 1 (Separability Test)

Let F and G be defined by (1). Multiply FG . Then

- (a) If $F(x)G(y) = f(x, y)$, then $y' = f(x, y)$ is **separable**.
- (b) If $F(x)G(y) \neq f(x, y)$, then $y' = f(x, y)$ is **not separable**.

Compute F and G in Relation $f(x, y) = F(x)G(y)$ _____

Assume

$$f(x, y) = 6xy + 8y - 15x - 20$$

We determine, without factorization talent, the separation formulas

$$f(x, y) = (3x + 4)(2y - 5).$$

Invent values $x_0 = 0$, $y_0 = 0$, chosen to make $f(x_0, y_0) = -20$ nonzero. Define

$$F(x) = \frac{f(x, y_0)}{f(x_0, y_0)} = \frac{0 + 0 - 15x - 20}{-20} = \frac{3}{4}x + 1,$$

$$G(y) = f(x_0, y) = 0 + 8y - 0 - 20.$$

Then $f(x, y) = F(x)G(y)$ because

$$F(x)G(y) = \left(\frac{3}{4}x + 1\right)(8y - 20) = 6xy + 8y - 15x - 20.$$

Non-Separability Tests

Test I. Equation $y' = f(x, y)$ is not separable provided for some pair of points (x_0, y_0) , (x, y) in the domain of f , (2) holds:

$$(2) \quad f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0.$$

Test II. The equation $y' = f(x, y)$ is not separable if either of the following conditions hold:

- $f_x(x, y) / f(x, y)$ is non-constant in y
- or
- $f_y(x, y) / f(x, y)$ is non-constant in x .

Test I details

Assume $f(x, y) = F(x)G(y)$, then equation (2) fails because each term on the left side of (2) equals $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y) (hence contradiction $0 \neq 0$).

Test II details

Assume $f(x, y) = F(x)G(y)$ and suppose F, G are sufficiently differentiable. Then

- $\frac{f_x(x, y)}{f(x, y)} = \frac{F'(x)}{F(x)}$ is independent of y

and

- $\frac{f_y(x, y)}{f(x, y)} = \frac{G'(y)}{G(y)}$ is independent of x .

Illustration

Consider $y' = xy + y^2$.

Test I implies it is not separable, because the left side of the relation is

$$\begin{aligned}\text{LHS} &= f(x, 1)f(0, y) - f(0, 1)f(x, y) \\ &= (x + 1)y^2 - (xy + y^2) \\ &= x(y^2 - y) \\ &\neq 0.\end{aligned}$$

Test II implies it is not separable, because

$$\frac{f_x}{f} = \frac{1}{x + y}$$

is not constant as a function of y .

Variables-Separable Method

The method determines two kinds of solution formulas.

Equilibrium Solutions.

They are the constant solutions $y = c$ of $y' = f(x, y)$. For any equation, find them by substituting $y = c$ into the differential equation.

Non-Equilibrium Solutions.

For a separable equation

$$y' = F(x)G(y),$$

a non-equilibrium solution y is a solution with $G(y) \neq 0$. It is found by dividing by $G(y)$, then applying the method of quadrature.

Theory of Non-Equilibrium Solutions

A given solution $\mathbf{y}(x)$ satisfying $\mathbf{G}(\mathbf{y}(x)) \neq \mathbf{0}$ throughout its domain of definition is called a non-equilibrium solution. Then division by $\mathbf{G}(\mathbf{y}(x))$ is allowed.

The *method of quadrature* applies to the separated equation $\mathbf{y}'/\mathbf{G}(\mathbf{y}(x)) = \mathbf{F}(x)$. Some details:

$$\int_{x_0}^x \frac{\mathbf{y}'(t) dt}{\mathbf{G}(\mathbf{y}(t))} = \int_{x_0}^x \mathbf{F}(t) dt$$

Integrate both sides of the separated equation over $x_0 \leq t \leq x$.

$$\int_{y_0}^{\mathbf{y}(x)} \frac{d\mathbf{u}}{\mathbf{G}(\mathbf{u})} = \int_{x_0}^x \mathbf{F}(t) dt$$

Apply on the left the change of variables $\mathbf{u} = \mathbf{y}(t)$. Define $\mathbf{y}_0 = \mathbf{y}(x_0)$.

$$\mathbf{y}(x) = \mathbf{M}^{-1} \left(\int_{x_0}^x \mathbf{F}(t) dt \right)$$

Define $\mathbf{M}(\mathbf{y}) = \int_{y_0}^{\mathbf{y}} d\mathbf{u}/\mathbf{G}(\mathbf{u})$. Take inverses to isolate $\mathbf{y}(x)$.

In practise, the last step with \mathbf{M}^{-1} is never done. The preceding formula is called the *implicit solution*. Some work is done to find algebraically an *explicit solution*, as is given by \mathbf{W}^{-1} .

Explicit and Implicit Solutions

Definition 1 (Explicit Solution)

A solution y of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y , followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y .

Definition 2 (Implicit Solution)

A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

Examples

- Explicit solutions: $y = 1$, $y = x$, $y = f(x)$, $y = 0$, $y = -1 + x^2$
- Implicit Solutions: $2y = 2$, $y^2 = x$, $y + x = 0$, $y = xy^2 + 1$, $y + 1 = x^2$, $x^2 + y^2 = 1$, $F(x, y) = c$

The General Solution of $y' = 2x(y - 3)$

- The variables-separable method gives equilibrium solutions $y = c$, which are already *explicit*. In this case, $y = 3$ is an equilibrium solution.
- Because $F = 2x$, $G = y - 3$, then division by G gives the quadrature-prepared equation $y'/(y - 3) = 2x$. A quadrature step gives the implicit solution

$$\ln |y - 3| = x^2 + C.$$

- The non-equilibrium solutions may be left in *implicit* form, giving the **general solution** as the list

$$L_1 = \{y = 3, \ln |y - 3| = x^2 + C\}.$$

- Algebra can be applied to $\ln |y - 3| = x^2 + C$ to write it as $y = 3 + ke^{x^2}$ where $k \neq 0$. Then general solution L_1 can be re-written as

$$L_2 = \{y = 3, y = 3 + ke^{x^2}\}.$$

List L_2 can be distilled to the single formula $y = 3 + ce^{x^2}$, but L_1 has no simpler expression.

Answer Check an Explicit Solution

To answer check $y' = 1 + y$ with **explicit solution** $y = -1 + ce^x$, expand the left side of the DE and the right side of the DE separately, then compare the two computations.

$$\text{LHS} = y'$$

$$= (-1 + ce^x)'$$

$$= 0 + ce^x$$

$$\text{RHS} = 1 + y$$

$$= -1 + 1 + ce^x$$

$$= ce^x$$

Left side of the DE.

Substitute the solution $y = -1 + ce^x$.

Evaluate.

Right side of the DE.

Substitute the solution $y = -1 + ce^x$.

Evaluate.

Then $\text{LHS} = \text{RHS}$ for all symbols. The DE is verified.

Answer Check an Implicit Solution

To answer check

$$y' = 1 + y^2$$

with **implicit solution**

$$\arctan(y) = x + c,$$

differentiate the implicit solution equation on x , to produce the differential equation.

$$\arctan(y(x)) = x + c$$

The implicit equation, replacing y by $y(x)$.

$$\frac{d}{dx} \arctan(y(x)) = \frac{d}{dx}(x + c)$$

Differentiate the previous equation.

$$\frac{y'(x)}{1 + (y(x))^2} = 1 + 0$$

Chain rule applied left.

$$y' = (1 + 0)(1 + y^2)$$

Cross-multiply to isolate y' left.

$$y' = 1 + y^2$$

The DE is verified.