Chapter 2

First Order Differential Equations

Contents

The subject of the chapter is the first order differential equation $y' = f(x, y)$. The study includes closed-form solution formulas for special equations, numerical solutions and some applications to science and engineering.

2.1 The Method of Quadrature

The **method of quadrature** refers to the technique of integrating both sides of an equation, hoping thereby to extract a solution formula. The name *quadrature* originates in geometry, where *quadrature* means *finding area*, a task overtaken in modern mathematics by *integration.* The naming convention is obeyed by `maple`, which lists it as its first method for solving differential equations. Below, Theorem 1, proved on page 71, isolates the requirements which make this method successful.

Theorem 1 (Quadrature)
Let $F(x)$ be continuous on $a < x < b$. Assume $a < x_0 < b$ and $-\infty < y_0 < \infty$. Then the initial value problem

$$y' = F(x), \quad y(x_0) = y_0 \tag{1}$$

has the unique solution

$$y(x) = y_0 + \int_{x_0}^{x} F(t)dt. \tag{2}$$

To apply the *method of quadrature* means: (i) Calculate a candidate solution formula by the *working rule* below; (ii) Verify the solution.

> To solve $y' = f(x, y)$ when f is independent of y, integrate on variable x across the equation.

River Crossing

A boat crosses a river at fixed speed with power applied perpendicular to the shoreline. Is it possible to estimate the boat's downstream location? The answer is *yes*. The problem's variables are

x	Distance from shore,	w	Width of the river,
y	Distance downstream,	v_b	Boat velocity (dx/dt),
t	Time in hours,	v_r	River velocity (dy/dt).

The calculus chain rule $dy/dx = (dy/dt)/(dx/dt)$ is applied, using the symbols v_r and v_b instead of dy/dt and dx/dt, to give the *model equation*

$$\frac{dy}{dx} = \frac{v_r}{v_b}. \tag{3}$$

Stream Velocity. The downstream river velocity will be approximated by $v_r = kx(w - x)$, where $k > 0$ is a constant. This equation gives velocity $v_r = 0$ at the two shores $x = 0$ and $x = w$, while the **maximum stream velocity** at the center $x = w/2$ is (see page 71)

$$v_c = \frac{kw^2}{4}. \tag{4}$$

Special River-Crossing Model. The model equation (3) using $v_r = kx(w-x)$ and the constant k defined by (4) give the initial value problem

$$\frac{dy}{dx} = \frac{4v_c}{v_b w^2} x(w - x), \quad y(0) = 0. \tag{5}$$

The solution of (5) by the method of quadrature is

$$y = \frac{4v_c}{v_b w^2}\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right), \tag{6}$$

where w is the river's width, v_c is the river's midstream velocity and v_b is the boat's velocity. In particular, the boat's **downstream drift** on the opposite shore is $\frac{2}{3}w(v_c/v_b)$. See *Technical Details* page 71.

Examples

1 **Example (Quadrature)** Solve $y' = 3e^x$, $y(0) = 0$.

Solution:

Candidate solution. The *working rule* is applied.

$y'(t) = 3e^t$	Copy the equation, x replaced by t.
$\int_0^x y'(t)dt = \int_0^x 3e^t dt$	Integrate across $0 \le t \le x$.
$y(x) - y(0) = 3e^x - 3$	Fundamental theorem of calculus, page 707.
$y(x) = 3e^x - 3$	Candidate solution found. Used $y(0) = 0$.

Verify solution. Let $y = 3e^x - 3$. The initial condition $y(0) = 0$ follows from $e^0 = 1$. To verify the differential equation, the steps are:

$\text{LHS} = y'$	Left side of the differential equation.
$= (3e^x - 3)'$	Substitute the expression for y.
$= 3e^x - 0$	Sum rule, constant rule and $(e^u)' = u'e^u$.
$= \text{RHS}$	Solution verified.

2 **Example (River Crossing)** A boat crosses a mile-wide river at 3 miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is 10 miles per hour. Find the transit time and the downstream drift to the opposite shore.

Solution: The answers, justified below, are 20 minutes and 20/9 miles.

Transit time. This is the time it takes to reach the opposite shore. The layman answer of 20 minutes is correct, because the boat goes 3 miles in one hour, hence 1 mile in 1/3 of an hour, perpendicular to the shoreline.

Downstream drift. This is the value $y(1)$, where y is the solution of equation (5), with $v_c = 10$, $v_b = 3$, $w = 1$, all distances in miles. The special model is

$$\frac{dy}{dx} = \frac{40}{3}x(1-x), \quad y(0) = 0.$$

The solution given by equation (6) is $y = \frac{40}{3}\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$ and the downstream drift is then $y(1) = 20/9$ miles. This answer is 2/3 of the layman's answer of (1/3)(10) miles; the explanation is that the boat is pushed downstream at a variable rate from 0 to 10 miles per hour.

Details and Proofs

Proof of Theorem 1:

Uniqueness. Let $y(x)$ be any solution of (1). It will be shown that $y(x)$ is given by the solution formula (2).

$y(x) = y(0) + \int_{x_0}^{x} y'(t)dt$	Fundamental theorem of calculus, page 707.
$= y_0 + \int_{x_0}^{x} F(t)dt$	Use (1).

Verification of the Solution. Let $y(x)$ be given by solution formula (2). It will be shown that $y(x)$ solves initial value problem (1).

$y'(x) = \left(y_0 + \int_{x_0}^{x} F(t)dt\right)'$	Compute the derivative from (2).
$= F(x)$	Apply the fundamental theorem of calculus.

The initial condition is verified in a similar manner:

$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t)dt$	Apply (2) with $x = x_0$.
$= y_0$	The integral is zero: $\int_a^a F(x)dx = 0$.

The proof is complete.

Technical Details for (4): The maximum of a continuously differentiable function $f(x)$ on $0 \le x \le w$ can be found by locating the critical points (i.e., where $f'(x) = 0$) and then testing also the endpoints $x = 0$ and $x = w$. The derivative $f'(x) = k(w - 2x)$ is zero at $x = w/2$. Then $f(w/2) = kw^2/4$. This value is the maximum of f, because $f = 0$ at the endpoints.

Technical Details for (6): Let $a = \dfrac{4v_c}{v_b w^2}$. Then

$y = y(0) + \int_0^x y'(t)dt$	Method of quadrature.
$= 0 + a\int_0^x t(w-t)dt$	By (5), $y' = at(w-t)$.
$= a\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right)$.	Integral table.

To compute the downstream drift, evaluate $y(w) = a\dfrac{w^3}{6}$ or $y(w) = \dfrac{2w}{3}\dfrac{v_c}{v_b}$.

Exercises 2.1

Quadrature. Find a candidate solution for each initial value problem and verify the solution. See Example 1, page 70.

1. $y' = 4e^{2x}$, $y(0) = 0$.

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1 + x)y' = x$, $y(0) = 0$.

4. $(1 - x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1+x^2)y' = 1$, $y(0) = 0$.

12. $(1+4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1+x)y' = 1$, $y(0) = 0$.

16. $(2+x)y' = 2$, $y(0) = 0$.

17. $(2+x)(1+x)y' = 2$, $y(0) = 0$.

18. $(2+x)(3+x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

20. $y' = (1+\cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing. A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 2, page 70, and the details for (6).

21. $w = 1$, $v_b = 4$, $v_c = 12$

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I. Verify the identity. Use the fundamental theorem of calculus part (b), page 707.

29. $\int_0^x (1+t)^3 dt = \frac{1}{4}\left((1+x)^4 - 1\right)$.

30. $\int_0^x (1+t)^4 dt = \frac{1}{5}\left((1+x)^5 - 1\right)$.

31. $\int_0^x te^{-t} dt = -xe^{-x} - e^{-x} + 1$.

32. $\int_0^x te^{t} dt = xe^{x} - e^{x} + 1$.

Fundamental Theorem II. Differentiate. Use the fundamental theorem of calculus part (b), page 707.

33. $\int_0^{2x} t^2 \tan(t^3) dt$.

34. $\int_0^{3x} t^3 \tan(t^2) dt$.

35. $\int_0^{\sin x} te^{t+t^2} dt$.

36. $\int_0^{\sin x} \ln(1+t^3) dt$.

Fundamental Theorem III. Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page 707. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x-1)$

38. $f(x) = x^2(x+1)$

39. $f(x) = \cos(3\pi x/4)$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1+x^2}$

42. $f(x) = \dfrac{2x}{1+x^4})$

43. $f(x) = x^2 e^{x^3}$

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1+x^2}}$

46. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

48. $f(x) = \dfrac{1}{\sqrt{1+4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

50. $f(x) = \dfrac{4x}{\sqrt{1-4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1+e^x}$

54. $f(x) = \dfrac{\ln |x|}{x}$

55. $f(x) = \sec^2 x$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x x$

60. $f(x) = \sec x \tan x x$

Integration by Parts. Integrate $\int_0^1 f(x)dx$ by parts, $\int u dv = uv - \int v du$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln |x|$

64. $f(x) = x \ln |x|$

65. $f(x) = x^2 e^{2x}$

66. $f(x) = (1+2x)e^{2x}$

67. $f(x) = x \cosh x$

68. $f(x) = x \sinh x$

69. $f(x) = x \arctan(x)$

70. $f(x) = x \arcsin(x)$

Partial Fractions. Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x+4}{x+5}$

72. $f(x) = \dfrac{x-2}{x-4}$

73. $f(x) = \dfrac{x^2+4}{(x+1)(x+2)}$

74. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)}$

75. $f(x) = \dfrac{x+4}{(x+1)(x+2)}$

76. $f(x) = \dfrac{x-1}{(x+1)(x+2))}$

77. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x+5)}$

78. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x+3)}$

79. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x-1)}$

80. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x-1)}$

Special Methods. Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2+2}{(x+1)^2}$, $u = x+1$.

82. $f(x) = \dfrac{x^2+2}{(x-1)^2}$, $u = x-1$.

83. $f(x) = \dfrac{2x}{(x^2+1)^3}$, $u = x^2+1$.

84. $f(x) = \dfrac{3x^2}{(x^3+1)^2}$, $u = x^3+1$.

85. $f(x) = \dfrac{x^3+1}{x^2+1}$, use long division.

86. $f(x) = \dfrac{x^4+2}{x^2+1}$, use long division.

Appendix A

Background Topics

Contents

Introduction

This introductory chapter contains a short list of topics that are extracted from pre-calculus and calculus courses.

A.1 Calculus

A small list of topics from differential and integral calculus are used in differential equations. The special notation of differential equations is introduced, along with some ideas of Isaac Newton, concerning the elementary kinetics formula $D = RT$, which has the physical interpretation $\mathsf{Distance} = \mathsf{Rate} \times \mathsf{Time}$.

Derivative

The calculus derivative $f'(x_0) = \lim_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ makes sense provided the indicated limit exists. Implicit in the formula is the *assumption* that f is defined in an open interval of the form $|x - x_0| < H$. Differential equations use this standard notation, plus the **Leibniz notation**

$$\frac{df}{dx} = f'(x).$$

Variable names used in science and engineering often follow this standard:

$$\begin{aligned} y &= \text{dependent variable}, \\ x &= \text{independent variable}. \end{aligned}$$

Within certain disciplines, such as kinetics, the variable names change, and the following standard exists:

$$\begin{aligned} x &= \text{displacement, dependent variable}, \\ t &= \text{time, independent variable}, \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= \text{velocity} & \frac{d^2x}{dt^2} &= \text{acceleration} \\ &= x'(t) & &= x''(t) \\ &= \dot{x}(t) & &= \ddot{x}(t) \\ &= Dx(t), & &= D^2x(t). \end{aligned}$$

The functional notation $y(x)$ means y is a dependent variable which depends on the independent variable x. For example, $x(t)$ means displacement x depends on time t. In a graphic, it is expected that x is the vertical axis and t is the horizontal axis. The dot-notation $\dot{x}(t)$ and $\ddot{x}(t)$, instead of $x'(t)$ and $x''(t)$, is common in literature on statics and dynamics. Operator notation Dx, D^2x appears in differential equations literature and in computer algebra systems, e.g., `maple` and `mathematica`.

Slope, Rates and Averages

The derivative can be interpreted geometrically as the **slope** of the line tangent to a curve at a point; see Figure 1.

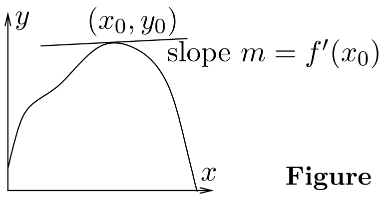

Figure 1. Slope of the tangent line.

The **tangent line** itself can be viewed as the **linearization** of the curve. For example, if the curve is the path of an automobile which at speedometer reading v instantly skids off the road, then the car follows the tangent line with constant speed v. Travel along the tangent line is **linear motion** at constant speed.

The line equation tangent to $y = f(x)$ at $x = x_0$ is given by the **point-slope form** of a line

$$\begin{aligned} &y - y_0 = m(x - x_0), \\ &y_0 = f(x_0), \quad m = f'(x_0). \end{aligned}$$

The notation $y(x)$, usual in differential equations, conflicts with the notation from geometry. In handwritten and blackboard work it is recommended to change x and y to capital letters X and Y, then replace f by y, as follows:

$$\begin{aligned} &Y - y_0 = m(X - x_0), \\ &y_0 = y(x_0), \quad m = y'(x_0). \end{aligned}$$

Other forms of a straight line in coordinate geometry are the **slope-intercept form** $y = mx + b$, the **standard form** $Ax + By + C = 0$ and the **parametric form**

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \quad -\infty < t < \infty. \end{cases}$$

In the parametric form, the vector $a\mathbf{i} + b\mathbf{j}$ is tangent to the line. For example, $a = 0$ and $b = 1$ gives a vertical line through (x_0, y_0).

Applied sciences interpret the derivative $f'(x)$ as the **rate of change** of $y = f(x)$ with respect to x. Typical interpretations appear below.

$\dot{x}(t)$	$\approx$	change in displacement x for a unit change in t
$\dfrac{dQ}{dt}$	$\approx$	change in charge Q for a unit change in t
$\ddot{Q}(t)$	$\approx$	change in current $I = \dot{Q}$ for a unit change in t
$A'(t)$	$\approx$	expected decrease in the amount A of radioactive material for time interval $[t, t+1]$

The **average** of n samples $y_1, \ldots, y_n$ is defined to be

$$\frac{y_1 + y_2 + \cdots + y_n}{n}.$$

The term **simple average** is sometimes used. The **average value** $\overline{f}$ of a continuous function $f(x)$ on $[a, b]$ is defined by

$$\overline{f} = \frac{\int_a^b f(x)dx}{b-a}.$$

This abstract notion has connections with the simple average. The theory of the integral $\int_a^b f(x)dx$ includes the **rectangular rule** for numerical integration (see also page **??**). For step size $h = (b-a)/n$ and sample values $y_1 = f(a)$, $y_2 = f(a+h)$, $\ldots$, $y_n = f(a+nh-h)$ it gives the approximation formula

$$\int_a^b f(x)dx \approx h(y_1 + y_2 + \cdots + y_n).$$

Multiply this relation by $1/(b-a)$ and replace the left side by the average value $\overline{f}$. Then

$$\overline{f} \approx \frac{y_1 + y_2 + \cdots + y_n}{n},$$

or in words,

> The average value $\overline{f}$ is approximately a simple average of n samples of f, taken at equi-spaced points in $[a, b]$.

In the language of kinetics, f is **velocity** and $\overline{f}$ is the **average velocity** or the **speed**.

The language of kinetics agrees with common public notions of speed. For example, the average of various speedometer reading samples during an automobile trip give a good indication of the average speed of the car on the trip. The average speed $R = \overline{f}$ is related to the trip time $T = b - a$ and the trip mileage D by the classical formula $D = RT$, which is taught in elementary school.

The expression for the trip mileage D in terms of the instantaneous velocity f,

$$D = \int_a^b f(x)dx,$$

is due to the creative genius of Isaac Newton. This relation of Newton today appears in texts as **the fundamental theorem of calculus**.

Fundamental Theorem of Calculus

The foundations of the study of differential equations rests with Newton's discovery of a way to state the relation $D = RT$ using instantaneous velocities instead of speed averages.

Theorem 1 (Fundamental theorem of calculus)
Let G be continuous and let F be continuously differentiable on $[a, b]$. Then

(a) $F(b) - F(a) = \int_a^b F'(x)dx$,

(b) $\frac{d}{dx}\int_a^x G(t)dt = G(x)$.

Part (a) of the fundamental theorem is used by calculus students to evaluate integrals. In differential equations, it is applied to find solutions.

Part (b) of the fundamental theorem computes the instantaneous rate of an averaging process. Calculus students use it to check answers to integration problems. In differential equations it is used to verify solutions.

The justification of $D = RT$ for instantaneous rates $f(x) = F'(x)$ is contained in part (a): divide both sides by $b - a$ and interpret the right side as *the average velocity* or *speed* to get the formula $D/T = R$.

1 **Example (Leibniz Notation)** Change $y''(x) + y(x)$ into Leibniz notation.

Solution:

$$y''(x) = \frac{d}{dx}y'(x) \qquad \text{Definition of second derivative.}$$
$$= \frac{d}{dx}\frac{dy}{dx} \qquad \text{Leibniz notation for the first derivative.}$$
$$= \frac{d^2y}{dx^2} \qquad \text{Leibniz notation.}$$

Therefore, the converted expression is $\frac{d^2y}{dx^2} + y$.

2 **Example (Notation Conversion)** Convert the equation $\frac{du}{dt} = u + e^t \sin t$ to dot notation.

Solution: By convention, $\frac{du}{dt} = \dot{u}(t)$ and $u = u(t)$. Therefore, the converted equation is $\dot{u}(t) = u(t) + e^t \sin t$.

3 **Example (Slope of the Tangent Line)** Compute the slope m of the line tangent to $y = x \sin x$ at $x = \pi/2$.

Solution:

$m = y'$	Definition of slope and derivative.
$= (x \sin x)'$	Definition of y.
$= \sin x + x \cos x$	Product rule and derivative tables. Variable x to be replaced by $\pi/2$.
$= \sin(\pi/2) + \dfrac{1}{2}\pi \cos(\pi/2)$	Replacement $x = \pi/2$.
$= 1$	Identities $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$ applied.

4 Example (Tangent Line Equation) Find the tangent line equation at $x = \pi/2$ for $y = x \sin x$ in point-slope form and in slope-intercept form.

Solution: The point-slope equation in an XY-system is $Y - y_0 = m(X - x_0)$. In this formula, $x_0 = \pi/2$, $y_0 = x_0 \sin x_0 = \pi/2$. Example 3 gives $m = 1$. The tangent line equation in point-slope form is $Y - \pi/2 = (1)(X - \pi/2)$, which simplifies to the slope-intercept form $Y = X$.

5 Example (Line Equations) Convert the line equation $y - 2 = 5(x - 3)$ to slope-intercept and parametric forms.

Solution: The *slope-intercept* form $y = 5x - 13$ is found by expansion to an explicit equation for y. A *parametric* form can be found by setting $x = t$ and then $y = 5x - 13 = 5t - 13$. The vector form is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 5t - 13 \end{pmatrix}, \quad -\infty < t < \infty.$$

6 Example (Decay Law Derivation) Derive the decay law $\dfrac{dA}{dt} = kA(t)$ from the sentence

> *Radioactive material decays at a rate proportional to the amount present.*

Solution: The sentence is first dissected into English phrases **1** to **4**.

1: *Radioactive material*	The phrase causes the invention of a symbol A for the amount present at time t.
2: *decays at a rate*	It means A undergoes decay. Then A changes. Calculus conventions imply the rate of change is dA/dt.
3: *proportional to*	Literally, it means *equal to a constant multiple of*. Let k be the proportionality constant.
4: *the amount present*	The amount of radioactive material present is $A(t)$.

The four phrases are translated into mathematical notation as follows.

Phrases 1 and 2	Symbol dA/dt.
Phrase 3	Equal sign '$=$' and a constant k.
Phrase 4	Symbol $A(t)$.

Let $A(t)$ be the amount present at time t. The translation is $\dfrac{dA}{dt} = kA(t)$.

7 Example (Average Value) Given $f(x) = xe^x + \sin^2(\pi x)$, find the average value on $0 \le x \le 2$.

Solution: The value is $\frac{1}{2}e^2 + 1$. The details:

$$\overline{f} = \frac{1}{2}\int_0^2 f(x)dx$$
Definition of average value, page 706.

$$= \frac{1}{2}\int_0^2 [xe^x + \sin^2(\pi x)]dx$$
Substitute for $f(x)$.

$$= \frac{1}{2}(x-1)e^x \Big|_{x=0}^{x=2} + \frac{1}{4\pi}\left(-\cos \pi x \sin \pi x + \pi x\right)\Big|_{x=0}^{x=2}$$
Integral tables.

$$= \frac{1}{2}e^2 + 1$$
Use $\sin(n\pi) = 0$.

8 Example (Speed) Find the speed for a car trip of 2 hours, given the velocity profile

$$\dot{x}(t) = \begin{cases} 1200t & 0 \le t \le 0.05, \\ 60 & 0.05 \le t \le 2. \end{cases}$$

Solution: The speed R is given by

$$R = \frac{1}{2}\int_0^2 \dot{x}(t)dt$$
Average value of $\dot{x}$, page 706.

$$= \frac{1}{2}\left(\int_0^{0.05} 1200t dt + \int_{0.05}^2 60 dt\right)$$
Use $\int_a^b f = \int_a^c f + \int_c^b f$.

$$= \frac{1}{2}\left(600(0.05)^2 + 60(2-0.05)\right)$$
Evaluate integrals.

$$= \frac{237}{4}.$$
About 59.25 mph.

The unrealistic 3-minute acceleration to 60 mph can be replaced by a more realistic 18-second acceleration to give 59.925 mph.

9 Example (Speed Estimation) Estimate the average speed of a car which accelerates from 0 to 65 miles per hour in 12 seconds.

Solution: The purpose of this example is to explain the layman's answer of 65/2 mph. The answer must be justified in the context of calculus.

If the acceleration is constant, then $\ddot{x}(t) = a =$ constant. Therefore, $\dot{x}(t) = at$, since $\dot{x}(0) = 0$. Let $t_0 = 12/3600$ hours. The average speed R for time interval $0 \le t \le t_0$ is

$R = \dfrac{1}{t_0}\displaystyle\int_0^{t_0} \dot{x}(t)dt$	Definition of average speed, page 706.
$= \dfrac{a}{t_0}\dfrac{t_0^2}{2}$	Evaluate integral with $\dot{x} = at$.
$= \dfrac{65}{2}$	Because $65 = \dot{x}(t_0) = at_0$.

It can be argued on physical grounds that no car has constant acceleration, so the answer 65/2 is merely an estimate. The layman's answer can be obtained by averaging the two speeds 0 and 65.

10 Example (Integral Identity) Verify the integral evaluation $\displaystyle\int_0^1 xe^x dx = 1$.

Solution:

$I = \displaystyle\int_0^1 xe^x dx$	Integral I to be evaluated.
$= \displaystyle\int_0^1 (xe^x - e^x)' \, dx$	Identity $xe^x = (xe^x - e^x)'$ derived below.
$= (xe^x - e^x)\big\vert_{x=0}^{x=1}$	Apply the fundamental theorem of calculus, part (a). See page 707.
$= 1$	Use $e^0 = 1$.

The identity $xe^x = (xe^x - e^x)'$ applied in the solution above is obtained by experiment, as follows.

$(xe^x)' = (1)e^x + xe^x$	Product rule $(uv)' = u'v + uv'$.
$= (e^x)' + xe^x$	Term xe^x isolated on the right.

Solving the last equation for xe^x gives the identity $xe^x = (xe^x - e^x)'$. A more systematic method for finding such identities is *integration by parts*.

11 Example (Integral Answer Check) Verify the identity

$$\int_0^x t\ln(1+t)dt = \frac{1}{2}\left(x^2 - 1\right)\ln(1+x) + \frac{x}{2} - \frac{x^2}{4}.$$

Solution: Both sides evaluate to zero at $x = 0$, because $\ln(1) = 0$. According to the fundamental theorem of calculus, part (b), page 707, it is sufficient to differentiate the answer on the right and verify that the derivative so obtained matches the integrand on the left. Let RHS denote the right hand side. Then

$\text{RHS}' = \left(\dfrac{x^2-1}{2}\ln(1+x) + \dfrac{x}{2} - \dfrac{x^2}{4}\right)'$	The *Right Hand Side* of the identity, to be differentiated.

$= x\ln(1+x) + \dfrac{x^2-1}{2x+2} + \dfrac{1}{2} - \dfrac{x}{2}$ — Product rule, power rule and the identity $(\ln(u))' = u'/u$.

$= x\ln(1+x).$ — Simplified derivative of the RHS.

The derivative of RHS matches the integrand of the left side, which completes the verification.

12 Example (Distance Estimate) Estimate the distance D traveled by an automobile in two hours, and its average speed R, given that for $t = 20$ to $t = 120$ the speedometer readings every 20 minutes are 55, 70, 66, 71, 72, 65 miles per hour.

Solution: The answers are 133 miles and 66.5 mph. To estimate the values of R and D, it will be assumed that the speed was constant during the 20-minute period before the reading. The actual velocity $\dot{x}(t)$ of the automobile is related to the average velocity R by the formula

$$R = \frac{1}{120}\int_0^{120} \dot{x}(t)dt.$$

The samples are used to find the average R as follows.

$R \approx \dfrac{55+70+66+71+72+65}{6}$ — Used $\overline{f} \approx \dfrac{y_1+\cdots+y_n}{n}$, page 706.

$= \dfrac{399}{6}$ — About 66.5 miles per hour.

Then $D = RT$ implies $D \approx \frac{399}{6}\frac{120}{60} = 133$ miles.

Exercises A.1

Derivative notation. Convert from the given notation, prime, dot, Leibniz or operator, to the other three forms.

1. $\dfrac{du}{dt}$

2. $\dot{u}(t_0)$

3. $\ddot{u}(1+t)$

4. $\dfrac{dx}{dt} = 1 + x(t)$

5. $D^2w(x) = 1 + w(x) + x$

6. $Dy(x) = y^{-2}(x)$

7. $\ln(w(r)) = \dfrac{dw}{dr}$

8. $e^{-y(x)} = y'(x)$

9. $\dot{y}(t) = 1 + t$

10. $\dot{x}(t) = e^{-2x(t)}$

Slope. Compute the slope of the line tangent to the curve at the given point.

11. $y = x^2 - 3x + 1, \quad x = 0.$

12. $y = x^5 - x + 2, \quad x = 2.$

13. $y = \sin x + x, \quad x = \pi/4.$

14. $y = \cos x - x, \quad x = \pi/4.$

15. $y = \tan^{-1} x + e^{-x}\ln(1+x), \quad x = 1.$

16. $y = \sin^{-1} x + e^{x}\ln(2+x), \quad x = 1.$

Tangent line equation. Find the tangent line equation in the three possible forms, point-slope, slope-intercept and parametric.

17. $y = x^3 - x, \quad x = 1.$

18. $y = x^3 + x + 1, \quad x = 0.$

19. $y = \sin^{-1}(x), \quad x = 1/2.$

20. $y = \tan^{-1}(x), \quad x = 1.$

21. $y = e^{-x}, \quad x = \ln(2).$

22. $y = \ln(1 + x), \quad x = 0.$

23. $y = \dfrac{1+x}{1-x}, \quad x = 0.$

24. $y = \dfrac{1-x^2}{1+x^2}, \quad x = 0.$

Rates. Model as a rate of change equation.

25. The expected change in charge Q is equal to the electromotive force $\sin(\omega t)$.

26. The damping force F is proportional to the instantaneous change in $x(t)$.

27. The angular rate of change is proportional to the external force $\cos(\omega t)$.

28. The amount in a bank account changes at a rate proportional to the current balance.

29. The expected population change is proportional to the present population P.

30. The temperature flux and the temperature difference from the surrounding medium are proportional.

Average value. Find the average value of f on $[a, b]$,

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)dx.$$

31. $xe^{-x}, \quad 0 \le x \le 1.$

32. $\dfrac{1}{2}e^x - \dfrac{1}{2}e^{-x}, \quad 0 \le x \le 2.$

33. $\ln x, \quad 1 \le x \le 3.$

34. $\sec x, \quad 0 \le x \le \pi/4.$

35. $x^3 - x, \quad 0 \le x \le 2.$

36. $\dfrac{x-1}{x+1}, \quad 0 \le x \le 1.$

37. $\dfrac{\sin x}{1 + \cos x}, \quad 0 \le x \le \pi/4.$

38. $\sin^3 x \cos x, \quad 0 \le x \le \pi.$

39. $\dfrac{1}{1+x^2}$ on $0 \le x \le 1/2$, $4/5$ on $1/2 \le x \le 1$.

40. $\dfrac{1}{x}$ on $1 \le x \le 2$, $\dfrac{5}{8}\dfrac{x^2}{1+x^2}$ on $2 \le x \le 3$.

41. $\tan x$ on $0 \le x \le \pi/4$, and $1 + (x - \pi/4)$ on $\pi/4 \le \pi/3$.

42. $\cot x$ on $\pi/4 \le x \le \pi/2$, and $x - \pi/2$ on $\pi/2 \le x \le \pi$.

Integral identities. Verify the given integration identity by applying the fundamental theorem of calculus.

43. $\displaystyle\int_0^1 \frac{1+t}{2+t}dt = 1 + \ln\frac{2}{3}.$

44. $\displaystyle\int_0^1 \frac{1+t^2}{2+t}dt = 5\ln\frac{3}{2} - \frac{3}{2}.$

45. $\displaystyle\int_0^\pi t\sin(2t)dt = \frac{\pi - 2}{4}.$

46. $\displaystyle\int_0^{\pi/2} t\cos(2t)dt = -\frac{1}{2}.$

47. $\displaystyle\int_0^1 te^{-t}dt = 1 - \frac{2}{e}.$

48. $\displaystyle\int_0^1 t^2e^{-t}dt = 2 - \frac{5}{e}.$

49. $\int_0^x \sin^4(t)\cos(t)dt = \frac{\sin^5(x)}{5}$.

50. $\int_0^x \tan(t)dt = -\ln(\cos x)$.

Car trip. Estimate the average speed R and the distance traveled D on a car trip, given the velocity samples.

51. Every 10 minutes from $t = 10$ to $t = 120$ minutes, 51, 62, 55, 53, 60, 67, 61, 67, 55, 70, 71, 66 miles per hour.

52. Every 15 minutes from $t = 15$ to $t = 225$ minutes, 90, 92, 110, 112, 120, 113, 109, 90, 95, 97, 60, 90, 100, 105, 103 kilometers per hour.

53. Every 5 minutes from $t = 5$ to $t = 75$ minutes, 45, 60, 61, 63, 60, 58, 61, 65, 25, 40, 45, 60, 65, 59, 60 miles per hour.

54. Every 5 minutes from $t = 5$ to $t = 100$ minutes, 50, 90, 100, 120, 110, 112, 130, 120, 110, 40, 60, 100, 90, 80, 20, 55, 130, 130, 120, 125 kilometers per hour.