1.6 Computing and Existence

The initial value problem

$$(1) \qquad y' = f(x, y), \quad y(x_0) = y_0$$

is studied here from a computational viewpoint. Answered are some basic questions about practical and theoretical computation of solutions:

- What in problem (1) causes numerical methods to fail?
- What hypotheses for (1) make numerical methods applicable?
- When does (1) have a symbolic solution?

Three Key Examples

The range of unusual behavior of solutions to $y' = f(x, y)$, $y(x_0) = y_0$ can be illustrated by three examples.

(A) $y' = 3(y-1)^{2/3}$, $y(0) = 1$.

The right side $f(x, y)$ is continuous. It has two solutions $y = 1 + x^3$ and $y = 1$.

(B) $y' = \dfrac{2y}{x-1}$, $y(0) = 1$.

The right side $f(x, y)$ is discontinuous. It has infinitely many piecewise-defined solutions

$$y = \begin{cases} (x-1)^2 & x < 1, \\ c(x-1)^2 & x \geq 1. \end{cases}$$

(C) $y' = 1 + y^2$, $y(0) = 0$.

The right side $f(x, y)$ is differentiable. It has unique solution $y = \tan(x)$, which satisfies $y(x_0) = \infty$ at time $x_0 = \pi/2$.

Numerical method failure can be caused by multiple solutions to problem (1), e.g., examples (A) and (B), because a numerical method is going to compute just *one* answer; see Example 32, page 63. Multiple solutions are often signaled by discontinuity of either f or its partial derivative f_y. In (A), the right side $3(y-1)^{2/3}$ has an infinite partial at $y = 1$, while in (B), the right side $2y/(x-1)$ is infinite at $x = 1$.

Simple jump discontinuities, or **switches**, appear in modern applications of differential equations. It is important therefore to allow $f(x, y)$ to be discontinuous, in a limited way, but multiple solutions must be avoided, e.g., example (B). An important success story in electrical engineering is circuit theory with periodic and piecewise-defined inputs. See Example 33, page 64.

Discontinuities of f or f_y in problem (1) should raise questions about the applicability of numerical methods. Exactly why there is not a precise and foolproof test to predict failure of a numerical method remains to be explained.

Theoretical solutions exist for problem (1), if $f(x, y)$ is *continuous*; see Peano's theorem, page 62. This solution may blow up in a finite interval, e.g., $y = \tan(x)$ in example (C); see Example 34, page 64.

No closed-form solution formula exists as a result of the basic theory. In part, this dilemma is due to the possibility of multiple solutions, if f is only continuous, e.g., example (A). Additional assumptions of a general nature do not seem to help: *there is in general no closed-form formula available for the solution.*

Exactly one theoretical solution exists in problem (1), provided $f(x, y)$ and $f_y(x, y)$ are continuous; see the Picard–Lindelöf theorem, page 62. The situation with numerical methods improves dramatically: the most popular methods are tractable.

Why Not "Put it on the computer?"

Typically, scientists and engineers rely upon computer algebra systems and numerical laboratories, e.g., `maple`, `mathematica` and `matlab`.

Computerization of differential equations constantly improves, with the advent of computer algebra systems and ever-improving numerical methods. Indeed, neither an advanced degree in mathematics nor a wizard's hat is required to query these systems for a closed-form solution formula. Many cases are checked systematically in a few seconds.

Fail-safe mechanisms usually do not exist for applying modern software to problem (1). The computer algebra system `maple` reports the "solution" $y = 1 + x^3$ for the problem $y' = 3(y - 1)^{2/3}$, $y(0) = 1$. But the obvious equilibrium solution $y = 1$ is unreported. The `maple` numeric solver silently accepts the same problem and solves for $y = 1$. To experience this, execute in `maple V Release 5.1` the code below.

```
de:=diff(y(x),x)=3*(y(x)-1)^(2/3):  ic:=y(0)=1:
dsolve({de,ic},y(x));                   # Symbolic sol
p:=dsolve({de,ic},y(x),numeric); p(1); # Numerical sol
```

There was an improvement in the report in later versions, e.g., version 10. In these versions, $y(x) = 1$ was reported for both. The inference for the maple user is that there is a unique solution, but the model has multiple solutions, making both reports incorrect.

Numerical instability is typically not reported by computer software. To understand the difficulty, consider the differential equation

$$y' = y - 2e^{-x}, \quad y(0) = 1.$$

The symbolic solution is $y = e^{-x}$. Attempts to solve the equation numerically will inevitably compute the nearby solutions $y = ce^x + e^{-x}$, where c is small. As x grows, the numerical solution grows like e^x, and $|y| \to \infty$. For example, `maple` computes $y(30) \approx -72557$, but $e^{-30} \approx 0.94 \times 10^{-13}$. In reality, the solution $y = e^{-x}$ cannot be computed. The `maple` code:

```
de:=diff(y(x),x)=y(x)-2*exp(-x):  ic:=y(0)=1:
p:=dsolve({de,ic},y(x),numeric):  p(30);
```

Mathematical model formulation, prior to using a computer, seems to be an essential skill which does not come in the colorfully decorated package from the software vendor. It is this creative skill that separates the practicing scientist from the person on the street who has enough money to buy a computer program.

Closed-Form Existence-Uniqueness Theory

The **closed-form existence-uniqueness theory** describes models

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2}$$

for which a closed-form solution is known, as an equation of some sort. The objective of the theory for first order differential equations is to obtain existence and uniqueness by exhibiting a solution formula. The mathematical literature which documents these models is too vast to catalog in a textbook. We discuss only the most popular models.

`Dsolve` Engine in `Maple`. This computer algebra system has an implementation for some specialized equations within the closed-form theory. Below are some of the equation types examined by `maple` for solving a differential equation using classification methods. Not everything tried by `maple` is listed, e.g., Lie symmetry methods, which are beyond the scope of this text.

Equation Type	Differential Equation
Quadrature	$y' = F(x)$
Linear	$y' + p(x)y = r(x)$
Separable	$y' = f(x)g(y)$
Abel	$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$
Bernoulli	$y' + p(x)y = r(x)y^n$
Clairaut	$y = xy' + g(y')$
d'Alembert	$y = xf(y') + g(y')$
Chini	$y' = f(x)y^n - g(x)y + h(x)$

Homogeneous	$y' = f(y/x)$, $\quad y' = y/x + g(x)f(y/x)$, $y' = (y/x)F(y/x^\alpha)$, $y' = F((a_1x + a_2y + a_3)/(a_4x + a_5y + a_6))$
Rational	$y' = P_1(x, y)/P_2(x, y)$
Ricatti	$y' = f(x)y^2 + g(x)y + h(x)$

Not every equation can be solved as written — restrictions are made on the parameters. Omitted from the above list are **power series methods** and differential equations with **piecewise-defined coefficients**. They are part of the closed-form theory but use specialized representations of solutions.

Special Equation Preview. The program here is to catalog a short list of first order equations and their known solution formulas. The formulas establish existence of a solution to the given initial value problem. They preview what is possible; details and examples appear elsewhere in the text. The issue of uniqueness is often routinely settled, as a separate issue, by applying the Picard-Lindelöf theorem. See Theorem 5, page 62, *infra*.

First Order Linear $y' + p(x)y = r(x)$ $y(x_0) = y_0$	Let p and r be continuous on $a < x < b$. Choose any (x_0, y_0) with $a < x_0 < b$. Then $y = y_0 e^{-\int_{x_0}^x p(t)dt} + \int_{x_0}^x r(t)e^{\int_t^x p(s)ds}dt$.
First Order Separable $y' = F(x)G(y)$ $y(x_0) = y_0$	Let $F(x)$ and $G(y)$ be continuous on $a < x < b$, $c < y < d$. Assume $G(y) \neq 0$ on $c < y < d$. Choose any (x_0, y_0) with $a < x_0 < b$, $c < y_0 < d$. Then $y(x) = W^{-1}\left(\int_{x_0}^x F(t)dt\right)$ where $W(Y) = \int_{y_0}^Y du/G(u)$.
First Order Analytic $y' = a(x)y + b(x)$ $y(x_0) = y_0$	Assume a and b have power series expansions in $\lvert x - x_0\rvert < h$. Then the power series $y(x) = \sum_{n=0}^\infty y_n(x - x_0)^n$ is convergent in $\lvert x - x_0\rvert < h$ and the coefficients y_n are found from the recursion $n!y_n = \left(\frac{d}{dx}\right)^{n-1}(a(x)y(x) + b(x))\Big\rvert_{x=x_0}$.

General Existence-Uniqueness Theory

The **general existence-uniqueness theory** describes the features of a differential equation model which make it possible to compute both theoretical and numerical solutions.

Modelers who create differential equation models generally choose the differential equation based upon the intuition gained from the *closed-form* *theory* and the *general theory*. Sometimes, modelers are lucky enough to refine a model to some known equation with closed-form solution. Other times, they are glad for just a numerical solution to the problem. In any case, they want a model that is *tested* and *proven* in applications.

General Existence Theory in Applications. For scientists and engineers, the results can be recorded as the following statement:

> In applications it is usually enough to require $f(x, y)$ and $f_y(x, y)$ to be continuous. Then the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ is a well-tested model to which classical numerical methods apply.

The general theory results to be stated are due to **Peano** and to **Picard-Lindelöf**. The techniques of proof require advanced calculus, perhaps graduate real-variable theory as well.

Theorem 4 (Peano)
Let $f(x, y)$ be continuous in a domain D of the xy-plane and let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ is one solution (many more might exist) of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Definition 7 (Picard Iteration)
Define the constant function $y_0(x) = y_0$ and then define by iteration

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

The sequence $y_0(x), y_1(x), \ldots$ is called the **sequence of Picard iterates** for $y' = f(x, y)$, $y(x_0) = y_0$. See Example 35 for computational details.

Theorem 5 (Picard-Lindelöf)
Let $f(x, y)$ and $f_y(x, y)$ be continuous in a domain D of the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a *unique* function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n\to\infty} y_n(x) = y(x)$$

is satisfied for $|x - x_0| < h$ by the Picard iterates $\{y_n\}$.

The **Picard iteration** is the replacement for a closed-form solution formula in the general theory, because the Picard-Lindelöf theorem gives the uniformly convergent infinite series *solution*

$$y(x) = y_0 + \sum_{n=0}^{\infty} (y_{n+1}(x) - y_n(x)) . \quad (3)$$

Well, yes, it is a solution formula, but from examples it is seen to be currently impractical. There is as yet no known practical solution formula for the general theory.

The condition that f_y be continuous can be relaxed slightly, to include such examples as $y' = |y|$. The replacement is the following.

Definition 8 (Lipschitz Condition)
Let $M > 0$ be a constant and f a function defined in a domain D of the xy-plane. A **Lipschitz condition** is the inequality $|f(x, y_1)-f(x, y_2)| \le M|y_1 - y_2|$, assumed to hold for all (x, y_1) and (x, y_2) in D. The most common way to satisfy this condition is to require the partial derivative $f_y(x, y)$ to be continuous (Exercises page 67).

Theorem 6 (Extended Picard-Lindelöf)
Let $f(x, y)$ be continuous and satisfy a Lipschitz condition in a domain D of the xy-plane. Let (x_0, y_0) belong to the interior of D. Then there is a small $h > 0$ and a *unique* function $y(x)$ continuously differentiable on $|x - x_0| < h$ such that $(x, y(x))$ remains in D for $|x - x_0| < h$ and $y(x)$ solves

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The equation

$$\lim_{n\to\infty} y_n(x) = y(x)$$

is satisfied for $|x - x_0| < h$ by the Picard iterates $\{y_n\}$.

32 **Example (Numerical Method Failure)** A project models $y = 1 + x^3$ as the solution of the problem $y' = 3(y - 1)^{2/3}$, $y(0) = 1$. Computer work gives the solution as $y = 1$. Is it bad computer work or a bad model?

Solution: It is a bad model, explanation to follow.

Solution verification. One solution of the initial value problem is given by the equilibrium solution $y = 1$. Another is $y = 1 + x^3$. Both are verified by direct substitution, using methods similar to Example 20, page 34.

Bad computer work? Technically, the computer made no mistake. Production numerical methods use only $f(x, y) = 3(y-1)^{2/3}$ and the initial condition $y(0) = 1$. They apply a fixed algorithm to find successive values of y. The algorithm is expected to be successful, that is, it will compute a list of data points which can be graphed by "connect-the-dots." The majority of numerical methods applied to this example will compute $y = 1$ for all data points.

Bad model? Yes, it is a bad model. The model does not *define* the solution $y = 1 + x^3$. The lesson here is that knowing a solution to an equation does not guarantee a numerical laboratory will be able to compute it.

Detecting a bad model. The right side $f(x, y)$ of the differential equation is continuous, but not continuously differentiable, therefore Picard's theorem does not apply, although Peano's theorem says a solution exists. Peano's theorem allows multiple solutions, but Picard's theorem does not. Sometimes, the only signal for non-uniqueness is the failure of application of Picard's theorem.

33 **Example (Switches)** The problem $y' = f(x, y)$, $y(0) = y_0$ with

$$f(x, y) = \begin{cases} 0 & 0 \le x \le 1, \\ 1 & 1 < x < \infty, \end{cases}$$

has a piecewise-defined discontinuous right side $f(x, y)$. Solve the initial value problem for $y(x)$.

Solution: The solution $y(x)$ is found by dissection of the problem according to the two intervals $0 \le x \le 1$ and $1 < x < \infty$ into the two differential equations $y' = 0$ and $y' = 1$. By the fundamental theorem of calculus, the answers are $y = y_0$ and $y = x + y_1$, where y_1 is to be determined. At the common point $x = 1$, the two solutions should agree (we ask y to be continuous at $x = 1$), therefore $y_0 = 1 + y_1$, giving the final solution

$$y(x) = \begin{cases} y_0 & \text{on } 0 \le x \le 1, \\ x - 1 + y_0 & \text{on } 1 < x < \infty. \end{cases}$$

The function $y(x)$ is continuous, but y' is discontinuous at $x = 1$. The differential equation $y' = f(x, y)$ and initial condition $y(0) = y_0$ are formally satisfied.

Is there another solution? No, because the method applied here assumed that $y(x)$ worked in the differential equation, and if it did, then it had to agree with $y = y_0$ on $0 \le x < 1$ and $y = x - 1 + y_0$ on $1 < x < \infty$, by the fundamental theorem of calculus.

Technically adept readers will find a flaw in the solution presented here, because of the treatment of the point $x = 1$, where y' does not exist. The flaw vanishes if we agree to verify the differential equation except at finitely many x-values where y' is undefined.

34 **Example (Finite Blowup)** Verify that $dx/dt = 2 + 2x^2$, $x(0) = 0$ has the unique solution $x(t) = \tan(2t)$, which approaches infinity in finite time $t = \pi/4$.

Solution: The function $x(t) = \tan(2t)$ works in the initial condition $x(0) = 0$, because $\tan(0) = \sin(0)/\cos(0) = 0$. The well-known trigonometric identity $\sec^2(2t) = 1 + \tan^2(2t)$ and the differentiation identity $x'(t) = 2\sec^2(2t)$ shows that the differential equation $x' = 2+2x^2$ is satisfied. The equation $x(\pi/4) = \infty$ is verified from $\tan(2t) = \sin(2t)/\cos(2t)$. Uniqueness follows from Picard's theorem, because $f(t, x) = 2+2x^2$ and $f_x(t, x) = 4x$ are continuous everywhere.

35 **Example (Picard Iterates)** Compute the Picard iterates $y_0, \ldots, y_3$ for the initial value problem $y' = f(x, y)$, $y(0) = 1$, given $f(x, y) = x - y$.

Solution: The answers are

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1 - x + x^2/2, \\
y_2(x) &= 1 - x + x^2 - x^3/6, \\
y_3(x) &= 1 - x + x^2 - x^3/3 + x^4/24.
\end{aligned}$$

The details for all computations are similar. A sample computation appears below.

$y_0 = 1$	This follows from $y(0) = 1$.
$y_1 = 1 + \int_0^x f(t, y_0(t))dt$	Use the formula with $n = 1$.
$= 1 + \int_0^x (t - 1)dt$	Substitute $y_0(x) = 1$, $f(x, y) = x - y$.
$= 1 - x + x^2/2$	Evaluate the integral.

The exact answer is $y = x - 1 + 2e^{-x}$. A Taylor series expansion of this function motivates why the Picard iterates converge to $y(x)$. See the exercises for details, page 67.

The `maple` code which does the various computations appears below. The code involving `dsolve` is used to compute the exact solution and the series solution.

```
y0:=x->1:f:=(x,y)->x-y:
y1:=x->1+int(f(t,y0(t)),t=0..x):
y2:=x->1+int(f(t,y1(t)),t=0..x):
y3:=x->1+int(f(t,y2(t)),t=0..x):
y0(x),y1(x),y2(x),y3(x);
de:=diff(y(x),x)=f(x,y(x)): ic:=y(0)=1:
dsolve({de,ic},y(x)); dsolve({de,ic},y(x),series);
```

Exercises 1.6

Multiple Solution Example. Define $f(x, y) = 3(y - 1)^{2/3}$. Consider $y' = f(x, y)$, $y(0) = 1$.

1. Do an answer check for $y(x) = 1$. Do a second answer check for $y(x) = 1 + x^3$.

2. Let $y(x) = 1$ on $0 \le x \le 1$ and $y(x) = 1 + (x - 1)^3$ for $x \ge 1$. Do an answer check for $y(x)$.

3. Does $f_y(x, y)$ exist for all (x, y)?

4. Verify that Picard's theorem does not apply to $y' = f(x, y)$, $y(0) = 1$, due to discontinuity of f_y.

5. Verify that Picard's theorem applies to $y' = f(x, y)$, $y(0) = 2$.

6. Let $y(x) = 1 + (x + 1)^3$. Do an answer check for $y' = f(x, y)$, $y(0) = 2$. Does another solution exist?

Discontinuous Equation Example. Consider $y' = \frac{2y}{x-1}$, $y(0) = 1$. Define $y_1(x) = (x-1)^2$ and $y_2(x) = c(x-1)^2$. Define $y(x) = y_1(x)$ on $-\infty < x < 1$ and $y(x) = y_2(x)$ on $1 < x < \infty$. Define $y(1) = 0$.

7. Do an answer check for $y_1(x)$ on $-\infty < x < 1$. Do an answer check for $y_2(x)$ on $1 < x < \infty$. Skip condition $y(0) = 1$.

8. Justify one-sided limits $y(1+) = y(1-) = 0$. The functions y_1 and y_2 join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

9. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

10. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Finite Blowup Example. Consider $y' = 1+y^2$, $y(0) = 0$. Let $y(x) = \tan x$.

11. Do an answer check for $y(x)$.

12. Find the partial derivative f_y for $f(x, y) = 1 + y^2$. Justify that f and f_y are everywhere continuous.

13. Justify that Picard's theorem applies, hence $y(x)$ is the only possible solution to the initial value problem.

14. Justify for $a = -\pi/2$ and $b = \pi/2$ that $y(a+) = -\infty$, $y(b-) = \infty$. Hence $y(x)$ blows up for finite values of x.

Numerical Instability Example. Let $f(x, y) = y - 2e^{-x}$.

15. Do an answer check for $y(x) = e^{-x}$ as a solution of the initial value problem $y' = f(x, y)$, $y(0) = 1$.

16. Do an answer check for $y(x) = ce^x + e^{-x}$ as a solution of $y' = f(x, y)$.

Multiple Solutions. Consider the initial value problem $y' = 5(y-2)^{4/5}$, $y(0) = 2$.

17. Do an answer check for $y(x) = 2$. Do a second answer check for $y(x) = 2 + x^5$.

18. Verify that the hypotheses of Picard's theorem fail to apply.

19. Find a formula which displays infinitely many solutions to $y' = f(x, y)$, $y(0) = 2$.

20. Verify that the hypotheses of Peano's theorem apply.

Discontinuous Equation. Consider $y' = \frac{y}{x-1}$, $y(0) = 1$. Define $y(x)$ piecewise by $y(x) = -(x-1)$ on $-\infty < x < 1$ and $y(x) = c(x-1)$ on $1 < x < \infty$. Leave $y(1)$ undefined.

21. Do an answer check for $y(x)$. The initial condition $y(0) = 1$ applies only to the domain $-\infty < x < 1$.

22. Justify one-sided limits $y(1+) = y(1-) = 0$. The piecewise definitions of $y(x)$ join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

23. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

24. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Picard Iteration. Find the Picard iterates y_0, y_1, y_2, y_3.

25. $y' = y + 1$, $y(0) = 2$

26. $y' = y + 1$, $y(0) = 2$

27. $y' = y + 1$, $y(0) = 0$

28. $y' = 2y + 1$, $y(0) = 0$

29. $y' = y^2$, $y(0) = 1$

30. $y' = y^2$, $y(0) = 2$

31. $y' = y^2 + 1$, $y(0) = 0$

32. $y' = 4y^2 + 4$, $y(0) = 0$

33. $y' = y + x$, $y(0) = 0$

34. $y' = y + 2x$, $y(0) = 0$

Picard Iteration and Taylor Series. Find the Taylor polynomial $P_n(x) = y(0) + y'(0)x + \cdots + y^{(n)}(0)x^n/n!$ and compare with the Picard iterates. Use a computer algebra system, if possible.

35. $y' = y$, $y(0) = 1$, $n = 4$, $y(x) = e^x$

36. $y' = 2y$, $y(0) = 1$, $n = 4$, $y(x) = e^{2x}$

37. $y' = x - y$, $y(0) = 1$, $n = 4$, $y(x) = -1 + x + 2e^{-x}$

38. $y' = 2x - y$, $y(0) = 1$, $n = 4$, $y(x) = -2 + 2x + 3e^{-x}$

Numerical Instability. Use a computer algebra system or numerical laboratory. Let $f(x, y) = y - 2e^{-x}$.

39. Solve $y' = f(x, y)$, $y(0) = 1$ numerically for $y(30)$.

40. Solve $y' = f(x, y)$, $y(0) = 1 + 0.0000001$ numerically for $y(30)$.

Closed–Form Existence. Solve these initial value problems using a computer algebra system.

41. $y' = y$, $y(0) = 1$

42. $y' = 2y$, $y(0) = 2$

43. $y' = 2y + 1$, $y(0) = 1$

44. $y' = 3y + 2$, $y(0) = 1$

45. $y' = y(y - 1)$, $y(0) = 2$

46. $y' = y(1 - y)$, $y(0) = 2$

47. $y' = (y - 1)(y - 2)$, $y(0) = 3$

48. $y' = (y - 2)(y - 3)$, $y(0) = 1$

49. $y' = -10(1 - y)$, $y(0) = 0$

50. $y' = -10(2 - 3y)$, $y(0) = 0$

Lipschitz Condition. Justify the following results.

51. The function $f(x, y) = x - 10(2 - 3y)$ satisfies a Lipschitz condition on the whole plane.

52. The function $f(x, y) = ax + by + c$ satisfies a Lipschitz condition on the whole plane.

53. The function $f(x, y) = xy(1 - y)$ satisfies a Lipschitz condition on $D = \{(x, y) : |x| \le 1, \quad |y| \le 1\}$.

54. The function $f(x, y) = x^2 y(a - by)$ satisfies a Lipschitz condition on $D = \{(x, y) : x^2 + y^2 \le R^2\}$.

55. If f_y is continuous on D and the line segment from y_1 to y_2 is in D, then $f(x, y_1) - f(x, y_2) = \int_{y_1}^{y_2} f_y(x, u)du$.

56. If f and f_y are continuous on a disk D, then f is Lipschitz with $M = \max_D\{|f_y(x, u)|\}$.