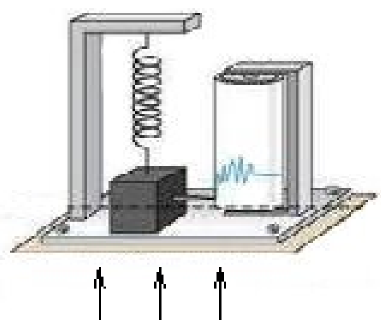
Math 2250, Lab 9 Extra Credit

References: Edwards-Penney Sections 5.3, 5.4 (Mechanical Vibrations) and 5.6 (Modeling Mechanical Systems). Course slides: *Unforced Oscillations* and *Forced Damped Vibrations*. For additional information on a mass-spring-dashpot system, and energy, visit the *Linear Physical Systems Analysis* web site of Erik Cheever, Professor of Engineering at Swarthmore College. The relevant section is Mechanical Systems (Translating), then the Mathematical Model and Energy/Power links.

1. **Energy in a mass-spring-dashpot system**
   The figure below depicts an example of a forced, damped mass-spring system. Illustrated in the figure is a seismoscope, a device used to record ground acceleration or displacement during an earthquake. The external force is the vertical ground force due to the earthquake.



In a damped mass-spring system, symbols m, c and k are non-negative constants representing respectively the mass in kilograms, the damping constant in Newton-seconds per meter, and the Hooke's constant in Newtons per meter. We use $x(t)$ to represent the signed displacement of the mass at time t, with $x = 0$ being the equilibrium position. Three forces act on the damped mass-spring system: the force $cx'(t)$ due to the dashpot, the Hooke's restoring force $kx(t)$ due to the spring, and the Newton's Second Law force mass$\times$acceleration $= mx''(t)$. The sum of the forces must equal the external forcing, $f(t)$. This gives the equation for a damped spring-mass system:

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

Suppose we wish to account for the total energy of the mass-spring configuration, neglecting the heat energy loss due to damping. We define the total energy $E(t)$ to be the sum of kinetic and potential energy. Potential energy $PE(t)$ is stored by the compressed or stretched spring, and it is the work done to stretch/compress the spring as the mass moves from equilibrium $x = 0$ to position $x(t)$:

$$PE(t) = \int_0^x (ku)du = \frac{k}{2}[x(t)]^2$$

Kinetic energy for translation of a mass m with velocity $v(t) = x'(t)$ is given by

$$KE(t) = \frac{m}{2}[x'(t)]^2.$$

The sum is the total energy

$$E(t) = PE(t) + KE(t) = \frac{1}{2}\left(k[x(t)]^2 + m[x'(t)]^2\right)$$

(a) Take the derivative of $E(t)$ with respect to time, using the chain rule on the right hand side of the equation. Then, simplify your result so that you get a formula for $E'(t)$ that only depends on the external forcing $f(t)$, the velocity $x'(t)$, and the damping coefficient c.

(b) Assume that $f(t) \equiv 0$ and the spring constant $k \neq 0$. In this case, what condition on the damping coefficient c guarantees that the energy in the system is constant (i.e., $dE/dt = 0$)?

(c) Solve the initial value problem $x'' + 2x' + 4x = 0$, $x(0) = 0$, $x'(0) = 1$. Substitute $x(t)$ and $x'(t)$ into the energy function $E(t)$ to obtain

$$E(t) = e^{-2t}\left(\frac{5}{6}\sin^2(\sqrt{3}t) - \frac{\sqrt{3}}{3}\sin(\sqrt{3}\,t)\cos(\sqrt{3}\,t) + \frac{1}{2}\cos^2(\sqrt{3}t)\right).$$

(d) Plot the energy curve $E(t)$ computed in part (c) on domain $0 \leq t \leq 4$.

(e) Answer the following questions about the energy plot in part (d).

1. What values were substituted for m, c, k, f to obtain the initial value problem in part (c)?
2. What is the physical meaning of the initial conditions $x(0) = 0, x'(0) = 1$?
3. Approximately how long will it take for the system in part (c) to lose 80% of its initial energy $E(0)$?
4. Give mathematical details for why the energy curve $E(t)$ in part (d) is monotonic.
5. The energy curve in part (d) is nearly constant on time intervals when the velocity is nearly zero. Explain in a sentence the contribution of potential and kinetic energy to $E'(t) \approx 0$ (meaning $E(t) \approx$ constant) on small time intervals.